

THE STABILIZER SUBGROUPS OF THE AUTOMORPHISM GROUP OF CERTAIN REAL EXCEPTIONAL JORDAN ALGEBRA.

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ABSTRACT. Let \mathcal{J}^1 be the real form of complex simple Jordan algebra with the automorphism group $F_{4(-20)}$. The stabilizer groups of $F_{4(-20)}$ -orbit on \mathcal{J}^1 are determined. As an application, for $F_{4(-20)}$, the Bruhat and Gauss decomposition, the Iwasawa decomposition and also the Iwasawa decomposition with respect to K_ϵ in the sense of T. Oshima and J. Sekiguchi are given concretely.

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0. INTRODUCTION AND OVERVIEW.

Let G be an exceptional linear Lie group of type F_4 defined by the automorphism group of an exceptional Jordan algebra. The objective of this paper is for $G = F_{4(-20)}$, to solve the following two problems:

(I) A classification of G -orbit:

(I.a) the decomposition of the space of elements in which G is represented, into equivalence classes or "orbit".

(I.b) the determining the Lie group structure of the stabilizer group for each G -orbits.

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(II) For G , giving concretely the Bruhat and Gauss decomposition, the Iwasawa decomposition and the Iwasawa decomposition with respect to K_ϵ (cf. [26]).

Solving two problems for $G = F_{4(-20)}$ and $K_\epsilon \cong \text{Spin}^0(8, 1)$ is basic, since its Lie algebra $\mathfrak{f}_{4(-20)}$ is rank one and the pair of Lie algebras $(\mathfrak{f}_{4(-20)}, \mathfrak{so}(8, 1))$ is split rank one (cf. [27]). The orbit decompositions are given not only for $G = F_{4(-20)}$ in [24], but also for $G = F_4^\mathbb{C}$ and $F_{4(4)}$ in [25]. In this paper, we will solve the problems (I.b) and (II) for $G = F_{4(-20)}$ and $K_\epsilon \cong \text{Spin}^0(8, 1)$.

The exceptional Jordan algebra \mathcal{J}^1 is defined in (1.3) with the Jordan product $X \circ Y$, and has the trace $\text{tr}(X)$, the non-definite but non-degenerate inner product $(X|Y)$, the identity element E of Jordan product, the cross product $X \times Y$, the determinant $\det(X)$ and the characteristic polynomial $\Phi_X(\lambda)$ (cf. (1.4), (1.5), (1.6), (1.11)). The linear Lie group $F_{4(-20)}$ is defined to be the automorphism group of the Jordan algebra \mathcal{J}^1 with Jordan product (cf. (1.12)). Then $F_{4(-20)}$ preserves the trace, the inner product, the identity element, the cross product, the determinant and the characteristic polynomial by Proposition 1.5. Then for the certain elements E_i , $F_i^1(x)$, P^+ , P^- , $Q^+(x)$ and $Q^-(y)$ (cf. (1.7), (1.8), (1.9)), we have:

Main Theorem 1. *The orbit types of $F_{4(-20)}$ -orbits on \mathcal{J}^1 are given as follows.*

(1) *Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 > \lambda_2 > \lambda_3$. Then X can be transformed to the following canonical forms by $F_{4(-20)}$ with the following type of stabilizer group.*

<i>The canonical forms of X</i>	<i>The type of stabilizer group</i>
1. $\text{diag}(\lambda_1, \lambda_2, \lambda_3)$	$\text{Spin}(8)$
2. $\text{diag}(\lambda_2, \lambda_3, \lambda_1)$	$\text{Spin}(8)$
3. $\text{diag}(\lambda_3, \lambda_1, \lambda_2)$	$\text{Spin}(8)$

(2) *Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ with $p \in \mathbb{R}$ and $q > 0$. Then X can be transformed to the following canonical form by $F_{4(-20)}$ with the following type of stabilizer group.*

<i>The canonical forms of X</i>	<i>The type of stabilizer group</i>
4. $\text{diag}(p, p, \lambda_1) + F_3^1(q)$	$\text{Spin}^0(7, 1)$

(3) *Assume that $X \in \mathcal{J}^1$ admits the characteristic roots λ_1 of multiplicity 1 and λ_2 of multiplicity 2. Then X can be transformed to the following canonical forms by $F_{4(-20)}$ with the following types of stabilizer group.*

<i>The canonical forms of X</i>	<i>The type of stabilizer group</i>
5. $\text{diag}(\lambda_1, \lambda_2, \lambda_2)$	$\text{Spin}(9)$
6. $\text{diag}(\lambda_2, \lambda_2, \lambda_1)$	$\text{Spin}^0(8, 1)$
7. $\text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+$	$\text{Spin}(7) \ltimes \text{Im}\mathbf{O}$
8. $\text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^-$	$\text{Spin}(7) \ltimes \text{Im}\mathbf{O}$

(4) Assume that $X \in \mathcal{J}^1$ admits the characteristic root of multiplicity 3. Then X can be transformed to the following canonical forms by $F_{4(-20)}$ with the following types of stabilizer group.

<i>The canonical forms of X</i>	<i>The type of stabilizer group</i>
9. $\frac{1}{3}\text{tr}(X)E$	$F_{4(-20)}$
10. $\frac{1}{3}\text{tr}(X)E + P^+$	$\text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})$
11. $\frac{1}{3}\text{tr}(X)E + P^-$	$\text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})$
12. $\frac{1}{3}\text{tr}(X)E + Q^+(1)$	$G_2 \ltimes (\text{Im}\mathbf{O} \times \text{Im}\mathbf{O})$

Let us denote $G = F_{4(-20)}$ and its Lie algebra as $\mathfrak{g} := \text{Lie}(G)$. For the certain element $\sigma_i \in G$ (cf. (3.2)) such that $\sigma_i \neq 1$ and $\sigma_i^2 = 1$ where 1 is denoted the identity element of G , let us denote involutive automorphisms $\tilde{\sigma}_i : \tilde{\sigma}_i(g) = \sigma_i g \sigma_i$ for $g \in G$. Then in Proposition 3.16(2)(3) and Lemma 6.2,

$$K := G^{\tilde{\sigma}_1} = G_{E_1} = \text{Spin}(9), \quad K_\epsilon = G^{\tilde{\sigma}_2} = G_{E_2} \cong \text{Spin}^0(8, 1).$$

Let us denote the differential of $\tilde{\sigma}_i$ also as $\tilde{\sigma}_i : \tilde{\sigma}_i\phi = \sigma_i\phi\sigma_i$ for $\phi \in \mathfrak{g}$. In Lemma 4.2(3), $\tilde{\sigma}_1$ is a Cartan involution. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition with respect to $\tilde{\sigma}_1$. Then $\mathfrak{k} = \text{Lie}(K)$. For certain element $\tilde{A}_3^1(1) \in \mathfrak{p}$ (cf. (3.1)), the 1-dimensional subspace \mathfrak{a} of \mathfrak{p} and the one parameter subgroup A are defined as $\mathfrak{a} = \{t\tilde{A}_3^1(1) \mid t \in \mathbb{R}\}$ and $A := \exp \mathfrak{a}$, respectively. The subgroup M of K and its Lie algebra are defined as $M := Z_K(\mathfrak{a})$ and $\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a})$, respectively. For the linear functional λ on \mathfrak{a} , let us denote $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \text{ for } H \in \mathfrak{a}\}$. Then in Lemma 4.5, there exists the linear functional α on \mathfrak{a} such that $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ and \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} . Here

$$\mathfrak{g}_\alpha = \{\mathcal{G}_1(x) \mid x \in \mathbf{O}\} \text{ and } \mathfrak{g}_{2\alpha} = \{\mathcal{G}_2(p) \mid p \in \text{Im}\mathbf{O}\}$$

(cf. (4.1) for the definition of $\mathcal{G}_i(x)$, Lemma 4.5(1)), and so any elements of \mathfrak{g}_α and $\mathfrak{g}_{2\alpha}$ are given by elements of \mathbf{O} (octonions) and $\text{Im}\mathbf{O}$ (vector part of octonions) (cf. (1.1), (1.2)), respectively. Put $\mathfrak{n}^+ := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, $\mathfrak{n}^- := \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$, $N^+ := \exp \mathfrak{n}^+$ and $N^- := \exp \mathfrak{n}^-$, respectively. Then N^+ and N^- are nilpotent subgroups of G such that $\tilde{\sigma}_1(N^\pm) = N^\mp$ (resp). Moreover, there exist the certain mappings $X \mapsto (X)_{E_i} \in \mathbb{R}$ and $X \mapsto (X)_{F_i^1} \in \mathbf{O}$ for $X \in \mathcal{J}^1$ (cf. (1.10)), and the mappings $\psi_1 : G \rightarrow \mathbf{O}$, $\psi_2 : G \rightarrow \text{Im}\mathbf{O}$ and $\psi_3 : G \rightarrow G$ (cf.

(5.3),(5.4),(5.5)). Then concrete construction of the Bruhat and Gauss decomposition of $F_{4(-20)}$ is given as follows.

Main Theorem 2.

(1) Assume that $g \in F_{4(-20)}$ and $(gP^+|P^-) \neq 0$. Let

$$\begin{aligned} t &:= -\frac{1}{2} \log \left(-\frac{(gP^+|P^-)}{4} \right) \in \mathbb{R}, \\ a_G(g) &:= \exp(t\tilde{A}_3^1(1)) \in A, \\ \bar{n}_G(g) &= \tilde{\sigma}_1 \left(\exp(-\mathcal{G}_1 \left(\frac{(\sigma_1 g^{-1} P^-)_{F_1^1} - \overline{(\sigma_1 g^{-1} P^-)_{F_2^1}}}{(gP^+|P^-)} \right) \right. \\ &\quad \left. - \mathcal{G}_2 \left(\frac{\text{Im}((\sigma_1 g^{-1} P^-)_{F_3^1})}{(gP^+|P^-)} \right) \right) \in N^-, \\ n_G(g) &:= \exp(t(\mathcal{G}_1(\psi_1(a_G(g)\bar{n}_G(g)g^{-1}) \\ &\quad + 2\mathcal{G}_2(\psi_2(a_G(g)\bar{n}_G(g)g^{-1})))) \in N^+, \\ m_G(g) &:= \psi_3(a_G(g)\bar{n}_G(g)g^{-1})^{-1}. \end{aligned}$$

Then

(i) $(gP^+|P^-) < 0$, and $a_G(g)$, $\bar{n}_G(g)$, $n_G(g)$, $m_G(g)$ are well-defined,

(ii) $m_G(g) \in M$ and

$$g = m_G(g)a_G(g)n_G(g)\bar{n}_G(g) \in MAN^+N^-.$$

(2) Assume $g \in F_{4(-20)}$ and $(gP^+|P^-) = 0$. Let

$$\begin{aligned} t &:= -\frac{1}{2} \log(-(gE_1|P^-)) \in \mathbb{R}, \\ a'(g) &= \exp(t\tilde{A}_3^1(1)) \in A, \\ n'(g) &:= \exp(t(\mathcal{G}_1(\psi_1(\sigma_1 a'(g)g^{-1}) + 2\mathcal{G}_2(\psi_2(\sigma_1 a'(g)g^{-1})))) \in N^+, \\ m'(g) &:= \psi_3(\sigma_1 a'(g)g^{-1})^{-1}. \end{aligned}$$

Then

(i) $(gE_1|P^-) < 0$, and $a'(g)$, $n'(g)$, $m'(g)$ are well-defined,

(ii) $m'(g) \in M$ and

$$g = m'(g)a'(g)n'(g)\sigma_1 \in MAN^+\sigma_1 = MAN^+\sigma_1N^-.$$

(3)

$$\begin{aligned} MAN^+N^- &= \{g \in F_{4(-20)} \mid (gP^+|P^-) \neq 0\} \\ &= \{g \in F_{4(-20)} \mid (gP^+|P^-) < 0\} \neq \emptyset, \\ MAN^+\sigma_1 &= MAN^+\sigma_1N^- \\ &= \{g \in F_{4(-20)} \mid (gP^+|P^-) = 0\} \neq \emptyset. \end{aligned}$$

Epecially,

$$\begin{aligned}
F_{4(-20)} &= MAN^+N^- \coprod MAN^+\sigma_1N^- \quad (\text{Bruhat decomposition}) \\
&= MAN^+N^- \coprod MAN^+\sigma_1 \\
(4) \quad &MAN^+N^- \text{ is open dense in } F_{4(-20)}. \text{ Especially} \\
F_{4(-20)} &= \overline{MAN^+N^-} \quad (\text{Gauss decomposition}).
\end{aligned}$$

The concrete construction of the Iwasawa decomposition of $F_{4(-20)}$ is given as follows (cf. Remark 8.10).

Main Theorem 3. *For any $g \in F_{4(-20)}$, let*

$$\begin{aligned}
H(g) &:= \frac{1}{2} \log(-(gP^-|E_1))\tilde{A}_3^1(1) \in \mathfrak{a}, \\
n(g) &:= \exp(\mathcal{G}_1 \left(\frac{(g^{-1}E_1)_{F_1^1} - \overline{(g^{-1}E_1)_{F_2^1}}}{(gP^-|E_1)} \right) \\
&\quad + \mathcal{G}_2 \left(\frac{\text{Im}((g^{-1}E_1)_{F_3^1})}{(gP^-|E_1)} \right)) \in N^+ \\
k(g) &:= gn(g)^{-1} \exp(-H(g)).
\end{aligned}$$

Then

- (1) $(gP^-|E_1) < 0$, and $H(g)$, $n(g)$, $k(g)$ are well-defined,
- (2) $k(g) \in K$ and

$$g = k(g)(\exp H(g))n(g) \in KAN^+.$$

The Iwasawa decomposition of $F_{4(-20)}$ with respect to K_ϵ in the sense of T. Oshima and J. Sekiguchi is given as follows.

Main Theorem 4. *Let \mathcal{D} be the domain of $F_{4(-20)}$ defined by*

$$\mathcal{D} := \{g \in F_{4(-20)} \mid (gP^-|E_2) > 0\}.$$

For any $g \in \mathcal{D}$, let

$$\begin{aligned}
H_\epsilon(g) &:= \frac{1}{2} \log((gP^-|E_2))\tilde{A}_3^1(1) \in \mathfrak{a}, \\
n_\epsilon(g) &:= \exp(\mathcal{G}_1 \left(\frac{(g^{-1}E_2)_{F_1^1} - \overline{(g^{-1}E_2)_{F_2^1}}}{(gP^-|E_2)} \right) \\
&\quad + \mathcal{G}_2 \left(\frac{\text{Im}((g^{-1}E_2)_{F_3^1})}{(gP^-|E_2)} \right)) \in N^+ \\
k_\epsilon(g) &:= gn_\epsilon(g)^{-1} \exp(-H_\epsilon(g)).
\end{aligned}$$

Then

(1) $k_\epsilon(g) \in K_\epsilon$ and

$$g = k_\epsilon(g)(\exp H_\epsilon(g))n_\epsilon(g) \in K_\epsilon AN^+,$$

(2) $\mathcal{D} = K_\epsilon AN^+ = \{g \in F_{4(-20)} \mid (gP^-|E_2) \neq 0\}$. Especially, \mathcal{D} is open dense in $F_{4(-20)}$.

There are two kinds of stabilizer groups of G . One is the kind of semisimple groups that are isomorphic to Spin groups. In Section 2, we explain the groups $\text{Spin}(8)$, $\text{Spin}(7)$ and G_2 by using the "triality". In Section 3, in Lemma 3.2, we show isomorphisms from $\text{Spin}(8)$ or $\text{Spin}(7)$ into G . Next we construct the stabilizer groups $\text{Spin}(9)$, $\text{Spin}^0(8, 1)$ and $\text{Spin}^0(7, 1)$, showing these groups are connected and two hold covering of $\text{SO}(9)$, $\text{O}^0(8, 1)$ and $\text{O}^0(7, 1)$, respectively.

The other kind of stabilizer groups of G is the kind of semidirect product type. In Proposition 5.7(2), we show

$$G_{P^-} = N^+M = MN^+.$$

At first, we show $M \cong \text{Spin}(7)$ in Proposition 4.4. At second, using the Campbell-Hausdorff-Dynkin formula and nilpotency, we determine the multiplication of N^+ and the operation of N^+ under \mathcal{J}^1 (cf. Lemma 4.9, Propositions 4.12 and 5.3(2)). At third, we construct the group $\text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})$ in (5.1) and Lemma 5.1. Then the multiplication of elements in N^+M makes the homomorphism $\varphi : \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O}) \rightarrow G_{P^-}$, and $\varphi(\text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})) = N^+M \subset G_{P^-}$ holds (cf. (5.2), Proposition 5.3(3)(4)). At last, in Section 5, we show that φ is an isomorphism and $G_{P^-} = N^+M$. For proving the surjectivity of φ , in Lemma 5.4, some orbits of some nilpotent subgroups are described by invariants under the group action (cf. Lemma 5.4). Furthermore, stabilizer groups which are semidirect product groups are isomorphic to some subgroups of N^+M (cf. Proposition 5.6). Using the orbit decomposition for G [24, Main Theorem] (cf. Proposition 1.10), we show Main Theorem 1.

In Section 7, since $G_{P^-} = N^+M$ and $G/G_{P^-} \simeq \mathcal{N}_1^-(\mathbf{O})$ (cf. (1.15)), considering AN^- -orbits on $\mathcal{N}_1^-(\mathbf{O})$, we show the concrete construction of Bruhat and Gauss decomposition in Main Theorem 2. Moreover in Theorem 7.6, we will show that the flag variety $G/(MAN^+)$ is obtained by a projectivization of an orbit $\mathcal{N}_1^-(\mathbf{O})$ of $F_{4(-20)}$.

In Section 8, since $G_{E_1} = K$ and $G/G_{E_1} \simeq \mathcal{H}(\mathbf{O})$ (cf. (1.13)), considering AN^+ -orbits on $\mathcal{H}(\mathbf{O})$, we show the concrete construction of Iwasawa decomposition in Main Theorem 3. In Remarks 8.9, we calculate the Gindikin-Karpelevich formula of G for c -function of Harish-Chandra.

In Section 9, since $G_{E_2} = K_\epsilon$ and $G/G_{E_2} \simeq \mathcal{H}'(\mathbf{O})$ (cf. (1.14)), considering AN^+ -orbits on $\mathcal{H}'(\mathbf{O})$, we show the concrete construction of Iwasawa decomposition with respect to K_ϵ in Main Theorem 4. And

in Theorem 9.6, we show a concrete construction of Matsuki decomposition [21, Theorem 3] of $F_{4(-20)}$.

Part I: Determining the group structure of the stabilizer groups.

1. PRELIMINARIES.

In this section, let n be a natural number, and p, q be non-negative integers with $p + q > 0$.

Let \mathbb{R} be the field of real numbers and $\mathbb{C} := \mathbb{R} \oplus \sqrt{-1}\mathbb{R}$ the field of complex numbers. Denote the cartesian n -power of a set X as $X^n := X \times \cdots \times X$ (n times). For $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , let V be a \mathbb{F} -linear space, $\mathrm{GL}_{\mathbb{F}}(V)$ the group of \mathbb{F} -linear automorphism of V , and $\mathrm{End}_{\mathbb{F}}(V)$ the linear space of \mathbb{F} -linear endomorphisms on V . A subset C is said to be a *cone* if $x \in V$ and $\lambda > 0$ imply that $\lambda x \in C$. The *exponential* of $f \in \mathrm{End}_{\mathbb{F}}(V)$ is defined by $\exp f = \sum_{k=0}^{\infty} \frac{f^k}{k!} \in \mathrm{GL}_{\mathbb{F}}(V)$. For a mapping $f : V \rightarrow V$, put $V_f := \{v \in V \mid fv = v\}$. Let G be a subgroup of $\mathrm{GL}_{\mathbb{F}}(V)$ and ϕ an automorphism on G and $v, v_i \in V$. Let us denote the subgroups of G as $G^{\phi} := \{g \in G \mid \phi g = g\}$, the *stabilizer* of v as $G_v := \{g \in G \mid gv = v\}$ and $G_{v_1, \dots, v_n} := \cap_{i=1}^n G_{v_i}$. Let us denote the G -orbit of v as $\mathrm{Orb}_G(v) := \{gv \mid g \in G\}$. For a \mathbb{R} -linear space V , its complexification $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \sqrt{-1}V$. For $f \in \mathrm{End}_{\mathbb{R}}(V)$, its complexification by $f^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(V^{\mathbb{C}})$ is written by the same letter f . The complex conjugation on $V^{\mathbb{C}}$ with respect to V is denoted by τ : $\tau(u + \sqrt{-1}v) := u - \sqrt{-1}v$ for all $u + \sqrt{-1}v \in V^{\mathbb{C}}$ with $u, v \in V$.

Let V be a \mathbb{F} -linear space. A *quadratic form* on V is a mapping $Q : V \rightarrow \mathbb{F}$ such that (i) $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in \mathbb{F}$ and $v \in V$, (ii) the associated symmetric form $B : V \times V \rightarrow \mathbb{F}$: $B(v, w) := \frac{1}{2}\{Q(v) + Q(w) - Q(v - w)\}$ is bilinear. The pair (V, Q) is called a *quadratic space*. Let $Q_{p,q}$ be a quadratic form on \mathbb{R}^{p+q} defined by $Q_{p,q}(x) := -(x_1^2 + \cdots + x_p^2) + (x_{p+1}^2 + \cdots + x_{p+q}^2)$ with $x = (x_1, \dots, x_{p+q})$, and denote the quadratic space by $(\mathbb{R}^{p,q}, Q_{p,q})$.

Let \mathbf{O} be the \mathbb{R} -algebra of *octonions* [9, 4, 40] with a base 1, $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and the multiplications among them are given as follows: 1 is the unit of \mathbb{R} ; $e_i^2 = -1$; $e_i e_j + e_j e_i = 0$ for $i \neq j$; $e_l e_m = e_n, e_m e_n = e_l$ and $e_n e_l = e_m$ for each $(l, m, n) \in \{(1, 2, 3), (3, 5, 6), (6, 7, 1), (1, 4, 5), (3, 4, 7), (6, 4, 2), (2, 5, 7)\}$. By convention, $e_0 := 1$. Let $\mathbf{O}^{\mathbb{C}} := \mathbf{O} \oplus \sqrt{-1}\mathbf{O}$ be the complexification of \mathbf{O} with the complex conjugation τ . Put $\tilde{\mathbf{O}} := \mathbf{O}$ or $\mathbf{O}^{\mathbb{C}}$. If $\tilde{\mathbf{O}} = \mathbf{O}$, then put $\mathbb{F} = \mathbb{R}$, and if $\tilde{\mathbf{O}} = \mathbf{O}^{\mathbb{C}}$, then $\mathbb{F} = \mathbb{C}$. For all $x = \sum_{i=0}^7 x_i e_i$ and $y = \sum_{i=0}^7 y_i e_i \in \tilde{\mathbf{O}}$ with $x_i, y_i \in \mathbb{F}$, the *conjugation*, the *inner product*, the *quadratic form*, the *vector part*

and the *scalar part* are defined as

$$\begin{aligned} \bar{x} &:= x_0 - \sum_{i=1}^7 x_i e_i, & (x|y) &:= \sum_{i=0}^7 x_i y_i, & \mathbf{n}(x) &:= (x|x), \\ (1.1) \quad \operatorname{Im}(x) &:= \frac{1}{2}(x - \bar{x}), & \operatorname{Re}(x) &:= \frac{1}{2}(x + \bar{x}) & (resp). \end{aligned}$$

Lemma 1.1. [9] (cf. [4], [37]) *Let $x, y, z, a, b \in \tilde{\mathbf{O}}$.*

- (1) $(xy|xy) = (x|x)(y|y)$.
- (2) $(ax|ay) = (a|a)(x|y) = (xa|ya)$.
- (3) $(ax|by) + (bx|ay) = 2(a|b)(x|y)$.
- (4) $(ax|y) = (x|\bar{a}y)$, $(xa|y) = (x|y\bar{a})$.
- (5) $\bar{\bar{x}} = x$, $\overline{x+y} = \bar{x} + \bar{y}$, $\overline{xy} = \bar{y} \bar{x}$.
- (6) $(x|y) = (y|x) = \frac{1}{2}(x\bar{y} + y\bar{x}) = \frac{1}{2}(\bar{x}y + \bar{y}x)$, $x\bar{x} = \bar{x}x = (x|x)$.
- (7) $\begin{cases} a(\bar{a}x) = (a\bar{a})x, a(x\bar{a}) = (ax)\bar{a}, x(a\bar{a}) = (xa)\bar{a}, \\ a(ax) = (aa)x, a(xa) = (ax)a, x(aa) = (xa)a. \end{cases}$
- (8) $\bar{b}(ax) + \bar{a}(bx) = 2(a|b)x = (xa)\bar{b} + (xb)\bar{a}$.
- (9) $(ax)y + x(ya) = a(xy) + (xy)a$, $(xa)y + (xy)a = x(ay) + x(ya)$,
 $(ax)y + (xa)y = a(xy) + x(ay)$.
- (10) $(ax)(ya) = a(xy)a$ (Moufang's formula).
- (11) $\operatorname{Re}(xy) = \operatorname{Re}(yx)$, $\operatorname{Re}(x(yz)) = \operatorname{Re}(y(zx)) = \operatorname{Re}(z(xy)) = \operatorname{Re}(xyz)$.

Let us denote

$$(1.2) \quad \operatorname{Im}\tilde{\mathbf{O}} := \{x \in \tilde{\mathbf{O}} \mid \operatorname{Re}(x) = 0\} = \{x \in \tilde{\mathbf{O}} \mid \bar{x} = -x\}.$$

Then the quadratic spaces (\mathbf{O}, \mathbf{n}) and $(\operatorname{Im}\mathbf{O}, \mathbf{n})$, are isomorphic to $(\mathbb{R}^{0,8}, Q_{0,8})$ and $(\mathbb{R}^{0,7}, Q_{0,7})$.

Let \mathbf{K} be a subalgebra of $\tilde{\mathbf{O}}$ such that $\bar{x} \in \mathbf{K}$ for all $x \in \mathbf{K}$, and $M(n, \mathbf{K})$ the set all $n \times n$ matrices with entries in \mathbf{K} . For $A \in M(n, \mathbf{K})$ with the (i, j) -entry $a_{ij} \in \mathbf{K}$, let ${}^t A \in M(n, \mathbf{K})$ be the transposed matrix having the (i, j) -entry a_{ji} , $\bar{A} \in M(n, \mathbf{K})$ the conjugate matrix having the (i, j) -entry \bar{a}_{ij} , and $A^* := {}^t \bar{A} \in M(n, \mathbf{K})$. Let us denote

$E := \operatorname{diag}(1, \dots, 1)$ and $I_p := \operatorname{diag}(\overbrace{-1, \dots, -1}^p, \overbrace{1, \dots, 1}^q) \in M(p+q, \mathbf{K})$. We consider the another field of complex numbers $\mathbf{C} := \{x_0 + x_1 e_1 \mid x_i \in \mathbb{R}\}$ which is the subalgebra of \mathbf{O} over \mathbb{R} . We use the following notations about some of classical Lie groups:

$$\begin{aligned} \mathrm{O}(n) &:= \{A \in M(n, \mathbb{R}) \mid {}^t A A = E\}, \\ \mathrm{SO}(n) &:= \{A \in M(n, \mathbb{R}) \mid {}^t A A = E, \det(A) = 1\}, \\ \mathrm{SU}(n) &:= \{A \in M(n, \mathbf{C}) \mid A^* A = E, \det(A) = 1\}, \\ \mathrm{O}(p, q) &:= \{A \in M(n, \mathbb{R}) \mid {}^t A I_p A = I_p\} \end{aligned}$$

where $\det(A)$ and $\text{tr}(A)$ denote the determinant of $A \in M(n, \mathbf{C})$ and the trace of $A \in M(n, \mathbf{C})$ over \mathbf{C} , respectively. For a Lie group G , its Lie algebra is denoted by $\text{Lie}(G)$. A connected Lie subgroup is often called an *analytic subgroup*. For topological spaces X and Y , $X \simeq Y$ denotes that X and Y are homeomorphic. For topological groups G and G' , $G \cong G'$ denotes that G and G' are isomorphic as topological group. Let G be a group with identity element 1. Then G^0 denotes the identity connected component. Let N be a normal subgroup of G and H a subgroup of G . If the equations $G = HN$ and $N \cap H = \{1\}$ hold, then G is called a *semidirect product of N and H* , and denote $G = H \ltimes N$.

For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in \mathbf{O}^3$, let us denote

$$h^1(\xi; x) := \begin{pmatrix} \xi_1 & \sqrt{-1}x_3 & \sqrt{-1}x_2 \\ \sqrt{-1}x_3 & \xi_2 & x_1 \\ \sqrt{-1}x_2 & \overline{x_1} & \xi_3 \end{pmatrix}.$$

The *exceptional Jordan algebra* \mathcal{J}^1 is defined as

$$(1.3) \quad \mathcal{J}^1 := \{h^1(\xi; x) \mid \xi \in \mathbb{R}^3, x \in \mathbf{O}^3\}$$

which has the *Jordan product* $X \circ Y$, the *inner product* $(X|Y)$ and the identity element E of Jordan product as

$$(1.4) \quad X \circ Y := \frac{1}{2}(XY + YX), \quad (X|Y) := \text{tr}(X \circ Y), \quad E := \text{diag}(1, 1, 1)$$

[36] (cf. [24], [35]). The *cross product* is defined by

$$(1.5) \quad X \times Y := \frac{1}{2}(2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - (X|Y))E)$$

[10] (cf. [16, p.232,(47)], [35], [32], [25]) with $X^{\times 2} := X \times X$ as well as an \mathbb{R} -linear form $(X|Y|Z) := (X \times Y|Z)$ and the *determinant*

$$(1.6) \quad \det(X) := \frac{1}{3}(X|X|X)$$

[11, p.163]. For all $X \in \mathcal{J}^1$, the *minimal subspace* V_X of X in \mathcal{J}^1 is defined by

$$V_X := \{aX^{\times 2} + bX + cE \mid a, b, c \in \mathbb{R}\}.$$

Then V_X is closed under the cross product ([25, Lemma 1.6(3)]). For $i \in \{1, 2, 3\}$ and $x \in \mathbf{O}$, let us denote

$$(1.7) \quad E_i = h^1(\delta_{i1}, \delta_{i2}, \delta_{i3}; 0, 0, 0), \quad F_i^1(x) := h^1(0, 0, 0; \delta_{i1}x, \delta_{i2}x, \delta_{i3}x)$$

where δ_{ij} is the Kronecker's delta,

$$(1.8) \quad P^+ := h^1(1, -1, 0; 0, 0, 1), \quad P^- := h^1(-1, 1, 0; 0, 0, 1),$$

$$(1.9) \quad Q^+(x) := h^1(0, 0, 0; x, \overline{x}, 0), \quad Q^-(x) := h^1(0, 0, 0; x, -\overline{x}, 0),$$

the subspaces $F_i^1(\mathbf{O}) := \{F_i^1(x) \mid x \in \mathbf{O}\}$, $F_i^1(\text{Im}\mathbf{O}) := \{F_i^1(p) \mid p \in \text{Im}\mathbf{O}\}$, $Q^+(\mathbf{O}) := \{Q^+(x) \mid x \in \mathbf{O}\}$ and $Q^-(\mathbf{O}) := \{Q^-(x) \mid x \in \mathbf{O}\}$ respectively. Then we have:

Lemma 1.2. (1) $\mathcal{J}^1 = \mathbb{R}E_1 \oplus \mathbb{R}E_2 \oplus \mathbb{R}E_3 \oplus F_1^1(\mathbf{O}) \oplus F_2^1(\mathbf{O}) \oplus F_3^1(\mathbf{O})$,
 (2) $\mathcal{J}^1 = \mathbb{R}(-E_1 + E_2) \oplus \mathbb{R}P^- \oplus \mathbb{R}E \oplus \mathbb{R}E_3 \oplus F_3^1(\text{Im}\mathbf{O}) \oplus Q^+(\mathbf{O}) \oplus Q^-(\mathbf{O})$.

Fix $X \in \mathcal{J}^1$. By Lemma 1.2(1),

$$X = h^1(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = \sum_{i=1}^3 (\xi_i E_i + F_i^1(x_i))$$

for some $\xi_i \in \mathbb{R}$ and $x_i \in \mathbf{O}$, then let us denote

$$(1.10) \quad (X)_{E_i} := \xi_i = (X|E_i), \quad (X)_{F_i^1} := x_i.$$

Moreover, by Lemma 1.2(2),

$$X = r(-E_1 + E_2) + sP^- + uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y)$$

for some $r, s, u, v \in \mathbb{R}$, $p \in \text{Im}\mathbf{O}$ and $x, y \in \mathbf{O}$, then let us denote

$$\begin{aligned} \{X\}_{-E_1+E_2} &:= r, & \{X\}_{P^-} &:= s, & \{X\}_E &:= u, & \{X\}_{E_3} &:= v, \\ \{X\}_{\text{Im}F_3^1} &:= p, & \{X\}_{Q^+} &:= x, & \{X\}_{Q^-} &:= y. \end{aligned}$$

We use the following notation in this paper:

$$\epsilon_i(j) := (-1)^{1+\delta_{ij}} \text{ where } \delta_{ij} \text{ is the Kronecker delta.}$$

Thus if $i = j$, then $\epsilon_i(j) = 1$, else $\epsilon_i(j) = -1$.

Lemma 1.3. ([25, Lemma 1.1, Lemma 1.6]) *Let $i, i+1, i+2, j \in \{1, 2, 3\}$ be counted modulo 3.*

(1) *Let $x, y \in \mathbf{O}$. Then*

$$\begin{cases} \text{(i)} & E_i \times E_i = 0, & \text{(ii)} & E_i \times E_{i+1} = \frac{1}{2}E_{i+2}, \\ \text{(iii)} & E_i \times F_i^1(x) = -\frac{1}{2}F_i^1(x), & \text{(iv)} & E_i \times F_j^1(x) = 0 \text{ (if } i \neq j), \\ \text{(v)} & F_i^1(x) \times F_i^1(y) = -\epsilon_1(i)(x|y)E_i, \\ \text{(vi)} & F_{i+1}^1(x) \times F_{i+2}^1(y) = -\epsilon_1(i)\frac{1}{2}F_i^1(\overline{xy}). \end{cases}$$

(2) *Let $X = \sum_{i=1}^3 (\xi_i E_i + F_i^1(x_i))$, $Y = \sum_{i=1}^3 (\eta_i E_i + F_i^1(y_i)) \in \mathcal{J}^1$.*

Then

$$\begin{cases} \text{(i)} & \text{tr}(X) = \sum_{i=1}^3 \xi_i, & \text{(ii)} & (X|Y) = \sum_{i=1}^3 (\xi_i \eta_i + \epsilon_1(i)2(x_i|y_i)), \\ \text{(iii)} & \det(X) = \xi_1 \xi_2 \xi_3 + 2\text{Re}((x_1 x_2) x_3) - \sum_{i=1}^3 \epsilon_1(i) \xi_i (x_i | x_i), \end{cases}$$

$$\text{(iv)} X^{\times 2} = \sum_{i=1}^3 ((\xi_{i+1} \xi_{i+2} - \epsilon_1(i)(x_i | x_i))E_i + F_i^1(-\epsilon_1(i)\overline{x_{i+1} x_{i+2}} - \xi_i x_i)).$$

$$(3) (X^{\times 2}) \circ X = \det(X)E, (X^{\times 2})^{\times 2} = \det(X)X.$$

Lemma 1.4. *Let $X \in \mathcal{J}^1$. Then*

$$\begin{cases} \text{(i)} & \{X\}_{-E_1+E_2} = \frac{1}{2}(P^-|X), & \text{(ii)} & \{X\}_{\text{Im}F_3^1} = \text{Im}((X)_{F_3^1}), \\ \text{(iii)} & \{X\}_{Q^+} = \frac{1}{2}((X)_{F_1^1} + \overline{(X)_{F_2^1}}), \\ \text{(vi)} & \{X\}_{Q^-} = \frac{1}{2}((X)_{F_1^1} - \overline{(X)_{F_2^1}}). \end{cases}$$

Proof. X can be expressed by $X = r(-E_1 + E_2) + sP^- + uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y)$ for some $r, s, u, v \in \mathbb{R}$, $p \in \text{Im } \mathbf{O}$ and $x, y \in \mathbf{O}$. (ii) follows immediately. By Lemma 1.3(2)(ii), $(P^-| -E_1 + E_2) = 2$ and $(P^-|P^-) = (P^-|E) = (P^-|E_3) = (P^-|F_3^1(p)) = (P^-|Q^+(x)) = (P^-|Q^-(y)) = 0$. Therefore $r = \frac{1}{2}(P^-|X)$. Last, (iii) and (vi) follows from $x + y = (X)_{F_1^1}$ and $\bar{x} - \bar{y} = (X)_{F_2^1}$. \square

For all $X \in \mathcal{J}^1$, let us denote $\varphi_X(\lambda) := \lambda E - X$. The *characteristic polynomial* $\Phi_X(\lambda)$ of $X \in \mathcal{J}^1$ is defined by

$$(1.11) \quad \Phi_X(\lambda) := \det(\varphi_X(\lambda)) = \lambda^3 - \text{tr}(X)\lambda^2 + \text{tr}(X^{\times 2})\lambda - \det(X)$$

(cf. [25]). The linear Lie group $F_{4(-20)}$ is defined by

$$(1.12) \quad F_{4(-20)} := \{g \in \text{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid g(X \circ Y) = gX \circ gY\}$$

(cf. [36, Theorem 2.2.2]).

Proposition 1.5. ([24, Theorem 1.4], cf. [25, Proposition 0.1(1)])

$$\begin{aligned} F_{4(-20)} &= \{g \in \text{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid g(X \times Y) = gX \times gY\} \\ &= \{g \in \text{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \det(gX) = \det(X), \ gE = E\} \\ &= \{g \in \text{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \det(gX) = \det(X), \ (gX|gY) = (X|Y)\} \\ &= \{g \in \text{GL}_{\mathbb{R}}(\mathcal{J}^1) \mid \Phi_{gX}(\lambda) = \Phi_X(\lambda)\} \\ &= \{g \in F_{4(-20)} \mid \text{tr}(gX) = \text{tr}(X)\}. \end{aligned}$$

A *characteristic root* of $X \in \mathcal{J}^1$ is said to be a solution of $\Phi_X(\lambda) = 0$ over \mathbb{C} . By Proposition 1.5, the trace, the inner product, the determinant, the identity element, the cross product and the characteristic polynomial are invariant under the action of $F_{4(-20)}$. Moreover the set of all characteristic roots and those multiplicities are invariant under the action of $F_{4(-20)}$.

Let us denote

$$(1.13) \quad \mathcal{H} := \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0, \ \text{tr}(X) = 1\},$$

$$(1.14) \quad \mathcal{H}(\mathbf{O}) := \{X \in \mathcal{H} \mid (X|E_1) \geq 1\},$$

$$(1.14) \quad \mathcal{H}'(\mathbf{O}) := \{X \in \mathcal{H} \mid (X|E_1) \leq 0\}.$$

Proposition 1.6. ([24, Proposition 1.6(1), Proposition 2.10(1)])

$$(1) \quad \mathcal{H} = \mathcal{H}(\mathbf{O}) \amalg \mathcal{H}'(\mathbf{O}).$$

$$(2) \quad \mathcal{H}(\mathbf{O}) = \text{Orb}_{F_{4(-20)}}(E_1).$$

$$(3) \quad \mathcal{H}'(\mathbf{O}) = \text{Orb}_{F_{4(-20)}}(E_2) = \text{Orb}_{F_{4(-20)}}(E_3).$$

A cone \mathcal{N} of \mathcal{J}^1 is defined by

$$\mathcal{N} := \{X \in \mathcal{J}^1 \mid \text{tr}(X) = \text{tr}(X^{\times 2}) = \det(X) = 0\}.$$

By Lemma 1.3(3), we observe that \mathcal{N} has the cones:

$$\begin{aligned} \mathcal{N}_1(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0, \operatorname{tr}(X) = 0, X \neq 0\}, \\ \mathcal{N}_1^+(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0, \operatorname{tr}(X) = 0, (X|E_1) > 0\}, \\ (1.15) \quad \mathcal{N}_1^-(\mathbf{O}) &:= \{X \in \mathcal{J}^1 \mid X^{\times 2} = 0, \operatorname{tr}(X) = 0, (X|E_1) < 0\}, \end{aligned}$$

and

$$\mathcal{N}_2(\mathbf{O}) := \{X \in \mathcal{J}^1 \mid \operatorname{tr}(X) = \operatorname{tr}(X^{\times 2}) = \det(X) = 0, X^{\times 2} \neq 0\}.$$

By convention, $\mathcal{N}_0(\mathbf{O}) := \{0\}$.

Lemma 1.7.

- (1) ([24, Proposition 1.6(2)]) $\mathcal{N}_1(\mathbf{O}) = \mathcal{N}_1^+(\mathbf{O}) \amalg \mathcal{N}_1^-(\mathbf{O})$.
- (2) $\mathcal{N} = \{0\} \amalg \mathcal{N}_1^+(\mathbf{O}) \amalg \mathcal{N}_1^-(\mathbf{O}) \amalg \mathcal{N}_2(\mathbf{O})$.

Proof. (2) Let $X \in \mathcal{N}$. By Lemma 1.3(3), $(X^{\times 2})^{\times 2} = \det(X)X = 0$. Thus $\mathcal{N} = \amalg_{i=0}^2 \mathcal{N}_i(\mathbf{O})$. Hence (2) follows from (1). \square

Proposition 1.8. ([24, Proposition 2.10(2), Proposition 4.3(4)])

- (1) $\mathcal{N}_1^+(\mathbf{O}) = \operatorname{Orb}_{F_4(-20)}(P^+)$.
- (2) $\mathcal{N}_1^-(\mathbf{O}) = \operatorname{Orb}_{F_4(-20)}(P^-)$.
- (3) $\mathcal{N}_2(\mathbf{O}) = \operatorname{Orb}_{F_4(-20)}(Q^+(1))$.

For $X \in \mathcal{J}^1$, and $\lambda_0 \in \mathbb{R}$, let us denote the elements of V_X as

$$\begin{aligned} p(X) &:= X - \frac{1}{3}\operatorname{tr}(X)E, \\ E_{X,\lambda_0} &:= \frac{1}{\operatorname{tr}((\lambda_0 E - X)^{\times 2})}(\lambda_0 E - X)^{\times 2}, \\ W_{X,\lambda_0} &:= X - (\lambda_0 E_{X,\lambda_0} + \frac{\operatorname{tr}(X) - \lambda_0}{2}(E - E_{X,\lambda_0})). \end{aligned}$$

Now, if E_{X,λ_1} is well-defined then

$$X = \lambda_0 E_{X,\lambda_0} + \frac{\operatorname{tr}(X) - \lambda_0}{2}(E - E_{X,\lambda_0}) + W_{X,\lambda_0}.$$

Proposition 1.9. ([24, Proposition 1.8]) For $X \in \mathcal{J}^1$, let λ_1 be a characteristic root of X in \mathbb{R} of multiplicity 1.

- (1) E_{X,λ_1} is well-defined (ie, $\operatorname{tr}((\lambda_1 E - X)^{\times 2}) \neq 0$), and $E_{X,\lambda_1} \in \mathcal{H}$.
- (2) For $g \in F_4(-20)$,

$$g(V_X) = V_{gX}, \quad gE_{X,\lambda_1} = E_{gX,\lambda_1}, \quad gW_{X,\lambda_1} = W_{gX,\lambda_1}.$$

Proposition 1.10. ([24, Main Theorem]) $F_4(-20)$ -orbits on \mathcal{J}^1 are classified as follows.

- (I) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 > \lambda_2 > \lambda_3$. Then there exists the unique $i \in \{1, 2, 3\}$ such that $\mathcal{H}(\mathbf{O}) \cap V_X = \{E_{X,\lambda_i}\}$ and $\mathcal{H}'(\mathbf{O}) \cap V_X = \{E_{X,\lambda_{i+1}}, E_{X,\lambda_{i+2}}\}$ where $i, i+1, i+2$ are

counted modulo 3. In this case, X can be transformed to one of the following canonical forms by $F_{4(-20)}$.

Cases	The canonical forms of X
1. $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$	$\text{diag}(\lambda_1, \lambda_2, \lambda_3)$
2. $E_{X,\lambda_2} \in \mathcal{H}(\mathbf{O})$	$\text{diag}(\lambda_2, \lambda_3, \lambda_1)$
3. $E_{X,\lambda_3} \in \mathcal{H}(\mathbf{O})$	$\text{diag}(\lambda_3, \lambda_1, \lambda_2)$

(II) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ ($q > 0$). Then X can be transformed to the following canonical form by $F_{4(-20)}$.

the characteristic roots of X	The canonical form of X
4. $\lambda_1 \in \mathbb{R}$, $p \pm \sqrt{-1}q$ ($q > 0$)	$\text{diag}(p, p, \lambda_1) + F_3^1(q)$

(III) Assume that $X \in \mathcal{J}^1$ admits the characteristic roots λ_1 of multiplicity 1 and λ_2 of multiplicity 2. Then $W_{X,\lambda_1} \in \mathcal{N}_1(\mathbf{O}) \amalg \{0\}$. In this case, X can be transformed to one of the following canonical forms by $F_{4(-20)}$.

Cases	The canonical form of X
5. $E_{X,\lambda_1} \in \mathcal{H}(\mathbf{O})$	$\text{diag}(\lambda_1, \lambda_2, \lambda_2)$
6. $E_{X,\lambda_1} \in \mathcal{H}'(\mathbf{O})$, $W_{X,\lambda_1} = 0$	$\text{diag}(\lambda_2, \lambda_2, \lambda_1)$
7. $W_{X,\lambda_1} \in \mathcal{N}_1^+(\mathbf{O})$	$\text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^+$
8. $W_{X,\lambda_1} \in \mathcal{N}_1^-(\mathbf{O})$	$\text{diag}(\lambda_2, \lambda_2, \lambda_1) + P^-$

(IV) Assume that $X \in \mathcal{J}^1$ admits the characteristic root of multiplicity 3. Then $p(X) \in \mathcal{N}$. In this case, X can be transformed to one of the following canonical forms by $F_{4(-20)}$.

Cases	The canonical form of X
9. $p(X) = 0$	$\frac{1}{3}\text{tr}(X)E$
10. $p(X) \in \mathcal{N}_1^+(\mathbf{O})$	$\frac{1}{3}\text{tr}(X)E + P^+$
11. $p(X) \in \mathcal{N}_1^-(\mathbf{O})$	$\frac{1}{3}\text{tr}(X)E + P^-$
12. $p(X) \in \mathcal{N}_2(\mathbf{O})$	$\frac{1}{3}\text{tr}(X)E + Q^+(1)$

(V) By $F_{4(-20)}$, the above canonical forms cannot be transformed from each other.

2. THE PRINCIPLE OF TRIALITY.

Put $S^0 := \{\text{diag}(1, 1, \dots, 1), \text{diag}(-1, 1, \dots, 1)\} \cong \mathbb{Z}_2$, and $\text{SO}(0) := \{1\}$ by convention. Then we have:

Lemma 2.1. *Let n be a natural number, and p, q non-negative integers with $p + q > 0$.*

- (1) $\text{O}(n) = S^0 \ltimes \text{SO}(n)$. Especially, $\text{O}^0(n) = \text{SO}(n)$.
- (2) $\text{O}^0(p, q) \simeq (\text{SO}(p) \times \text{SO}(q)) \times \mathbb{R}^{pq}$.
- (3) For all $n \geq 3$, the fundamental group $\pi_1(\text{O}^0(n, 1)) = \mathbb{Z}_2$

From now on, the groups $O(8)$, $SO(8)$, $O(7)$ and $SO(7)$ are identified with the groups

$$\begin{aligned} O(8) &= \{g \in GL_{\mathbb{R}}(\mathbf{O}) \mid (gx|gy) = (x|y)\}, \\ SO(8) &= \{g \in GL_{\mathbb{R}}(\mathbf{O}) \mid (gx|gy) = (x|y), \det(g) = 1\}, \\ O(7) &= \{g \in O(8) \mid g1 = 1\}, \\ SO(7) &= \{g \in SO(8) \mid g1 = 1\} \end{aligned}$$

respectively. $\epsilon \in O(8)$ is defined by

$$\epsilon x := \bar{x} \quad \text{for } x \in \mathbf{O}.$$

Then $\epsilon^2 = 1$ and its determinant is -1 : $\det(\epsilon) = -1$. The involutive automorphism t of the group $SO(8)^3$ is defined by

$$t(g_1, g_2, g_3) := (g_1, g_2, \epsilon g_3 \epsilon) \quad \text{for } (g_1, g_2, g_3) \in SO(8)^3.$$

The subgroup $T(\mathbf{O})$ of $SO(8)^3$ is defined by

$$T(\mathbf{O}) := \{(g_1, g_2, g_3) \in SO(8)^3 \mid (g_1 x)(g_2 y) = g_3(xy) \text{ for all } x, y \in \mathbf{O}\}$$

(cf. [2], [9, (2.4.6)], [22], [32], [40]), and the subgroup \tilde{D}_4 of $SO(8)^3$ by

$$\begin{aligned} \tilde{D}_4 &:= t^{-1}(T(\mathbf{O})) = \{(g_1, g_2, g_3) \in SO(8)^3 \mid t(g_1, g_2, g_3) \in T(\mathbf{O})\} \\ &= \{(g_1, g_2, g_3) \in SO(8)^3 \mid (g_1 x)(g_2 y) = \epsilon g_3 \epsilon(xy) \text{ for all } x, y \in \mathbf{O}\}. \end{aligned}$$

The equation $(g_1 x)(g_2 y) = g_3(xy)$ is called the *triality*.

Lemma 2.2. (Y. Matsushima [22], cf. [40, Lemma 1.14.3]) *Let indices $i, i+1, i+2$ be counted modulo 3. Assume that there exists $(g_1, g_2, g_3) \in O(8)^3$ such that $(g_i x)(g_{i+1} y) = \epsilon g_{i+2} \epsilon(xy)$ for all $x, y \in \mathbf{O}$. Then $(g_{i+1} x)(g_{i+2} y) = \epsilon g_i \epsilon(xy)$ for all $x, y \in \mathbf{O}$. Especially,*

$$(g_i, g_{i+1}, g_{i+2}) \in \tilde{D}_4 \Leftrightarrow (g_{i+1}, g_{i+2}, g_i) \in \tilde{D}_4.$$

For $i \in \{1, 2, 3\}$, the homomorphism $p_i : \tilde{D}_4 \rightarrow SO(8)$ is defined by

$$p_i(g_1, g_2, g_3) := g_i \quad \text{for } (g_1, g_2, g_3) \in \tilde{D}_4.$$

Lemma 2.3. *Let $x, y \in \mathbf{O}$.*

(1) *For all $g \in SO(7)$, the following equations hold.*

(i) $\overline{gx} = g\overline{x}$, (ii) $\epsilon g \epsilon = g$, (iii) $g(\text{Im}(x)) = \text{Im}(gx)$, (iv) $g(\text{Im}\mathbf{O}) \subset \text{Im}\mathbf{O}$.

(2) *Let $(g_1, g_2, g_3) \in \tilde{D}_4$. Then*

$$g_i 1 = 1 \text{ for some } i \in \{1, 2, 3\} \Leftrightarrow g_{i+1} = \epsilon g_{i+2} \epsilon \Leftrightarrow g_{i+2} = \epsilon g_{i+1} \epsilon$$

where indices $i, i+1, i+2$ are counted modulo 3.

(3) *Assume that $(g_1, g_2, g_3) \in \tilde{D}_4$ and $g_3 1 = 1$. Then*

$$\begin{aligned} \text{(i)} \quad & g_3(x\overline{y}) = (g_1 x)(\overline{g_1 y}), \quad \text{(ii)} \quad g_3(\text{Im}(x\overline{y})) = \text{Im}((g_1 x)(\overline{g_1 y})), \\ \text{(iii)} \quad & g_1(xy) = (g_3 x)(g_1 y). \end{aligned}$$

Proof. (1) (i) follows from $g\bar{x} = g(2(1|x) - x) = 2(1|gx) - gx = \overline{gx}$. By (i), $\epsilon g \epsilon x = g(\bar{x}) = gx$, and so (iii) follows from $g(\text{Im}(x)) = g(x - (1|x)) = gx - (1|gx) = \text{Im}(gx)$. For all $p \in \text{Im } \mathbf{O}$, by (iii), $gp = g(\text{Im}(p)) = \text{Im}(gp)$. Hence (iv) follows.

(2) Obviously $g_{i+1} = \epsilon g_{i+2} \epsilon$ iff $g_{i+2} = \epsilon g_{i+1} \epsilon$. Thus we will show $g_i 1 = 1$ iff $g_{i+1} = \epsilon g_{i+2} \epsilon$. Suppose that $g_i 1 = 1$. By Lemma 2.2, $(g_i, g_{i+1}, g_{i+2}) \in \tilde{D}_4$ and so $(g_i x)(g_{i+1} y) = \epsilon g_{i+2} \epsilon (xy)$. Putting $x = 1$, $\epsilon g_{i+2} \epsilon = g_{i+1}$. Conversely, suppose that $\epsilon g_{i+2} \epsilon = g_{i+1}$. Because of $(g_i, g_{i+1}, g_{i+2}) \in \tilde{D}_4$, $(g_i x)(\epsilon g_{i+2} \epsilon y) = (g_i x)(g_{i+1} y) = \epsilon g_{i+2} \epsilon (xy)$. Putting $x = y = 1$, $(g_i 1)(\epsilon g_{i+2} \epsilon 1) = \epsilon g_{i+2} \epsilon 1$. Multiplying $(\epsilon g_{i+2} \epsilon 1)^{-1}$ from right, $g_i 1 = 1$. Hence (2) follows.

(3) By (2), note that $g_2 = \epsilon g_1 \epsilon$ and $g_1 = \epsilon g_2 \epsilon$. Since $g_3 \in \text{SO}(7)$ and (1)(ii), $g_3 = \epsilon g_3 \epsilon$. Thus because of $(g_1, \epsilon g_1 \epsilon, g_3) \in \tilde{D}_4$, $g_3(x\bar{y}) = \epsilon g_3 \epsilon (x\bar{y}) = (g_1 x)(\epsilon g_1 \epsilon \bar{y}) = (g_1 x)(\overline{g_1 y})$. Hence (i) follows. Since $g_3 \in \text{SO}(7)$, (1)(iii) and (i), $g_3(\text{Im}(x\bar{y})) = \text{Im}(g_3(x\bar{y})) = \text{Im}((g_1 x)(\overline{g_1 y}))$. Hence (ii) follows. Last, by Lemma 2.2, $(g_3, g_1, g_2) \in \tilde{D}_4$. Thus because of $g_1 = \epsilon g_2 \epsilon$, $g_1(xy) = \epsilon g_2 \epsilon (xy) = (g_3 x)(g_1 y)$. Hence (iii) follows. \square

The subgroup \tilde{B}_3 of \tilde{D}_4 is defined as

$$\tilde{B}_3 := \{(g_1, g_2, g_3) \in \tilde{D}_4 \mid g_3 1 = 1\} = \{(g_1, g_2, g_3) \in \tilde{D}_4 \mid g_2 = \epsilon g_1 \epsilon\}.$$

and the homomorphism $q : \tilde{B}_3 \rightarrow \text{SO}(7)$ as $q := p_3|_{\tilde{B}_3}$: $q(g_1, g_2, g_3) = g_3$. The linear Lie group G_2 is defined by

$$G_2 := \text{Aut}(\mathbf{O}) = \{g \in \text{GL}_{\mathbb{R}}(\mathbf{O}) \mid (gx)(gy) = g(xy)\}.$$

For $g \in G_2$, putting $x = y = 1$, $g 1 = 1$, and so G_2 is a subgroup of $\text{SO}(7)$. For any $g \in G_2$, considering $(g, g, g) \in \text{SO}(8)^3$, G_2 is a subgroup of \tilde{B}_3 . Now $S^7 = \{a \in \mathbf{O} \mid \text{n}(a) = 1\}$ and $S^6 = \{a \in \text{Im } \mathbf{O} \mid \text{n}(a) = 1\}$. For all $a \in S^7$, the elements $L_a, R_a, T_a \in \text{GL}_{\mathbb{R}}(\mathbf{O})$ are defined as

$$L_a x := ax, \quad R_a x := xa, \quad T_a x := axa \quad \text{for } x \in \mathbf{O}$$

respectively. Since Lemma 1.1(2) and S^7 is connected, $L_a, R_a, T_a \in \text{SO}(8)$. For any $a_i \in S^7$, let us denote

$$L_{a_n, \dots, a_1} := L_{a_n} \cdots L_{a_1}, \quad R_{a_n, \dots, a_1} := R_{a_n} \cdots R_{a_1}, \quad T_{a_n, \dots, a_1} := T_{a_n} \cdots T_{a_1}.$$

Lemma 2.4.

(1) (cf. [40, Theorem 1.9.1, Theorem 1.9.2])

$$S^6 = \text{Orb}_{G_2}(e_1), \quad (G_2)_{e_1} \cong \text{SU}(3), \quad G_2/\text{SU}(3) \simeq S^6.$$

Furthermore, G_2 is connected.

(2) If $a_i \in S^7$, then $(L_{a_n, \dots, a_1}, R_{a_n, \dots, a_1}, \epsilon T_{a_n, \dots, a_1} \epsilon) \in \tilde{D}_4$.

(3) If $a, b \in S^6$, then $(L_{b,a}, R_{b,a}, T_{b,a}) \in \tilde{B}_3$.

(4) $G_2 = \{(g_1, g_2, g_3) \in \tilde{B}_3 \mid g_1 1 = 1\} = \{(g_1, g_2, g_3) \in \tilde{B}_3 \mid g_2 1 = 1\}$.

Proof. (2) Let $a \in S^7$. Using Lemma 1.1(10),

$$(L_a x)(R_a y) = T_a(xy) = \epsilon(\epsilon T_a \epsilon)\epsilon(xy).$$

Because of $L_a, R_a, \epsilon T_a \epsilon \in \text{SO}(8)$, $(L_a, R_a, \epsilon T_a \epsilon) \in \tilde{D}_4$. By induction,

$$(L_{a_n, \dots, a_1} x)(R_{a_n, \dots, a_1} y) = \epsilon(\epsilon T_{a_n, \dots, a_1} \epsilon)\epsilon(xy).$$

Hence (2) follows.

(3) By (2), $(L_{b,a}, R_{b,a}, \epsilon T_{b,a} \epsilon) \in \tilde{D}_4$. Since $a^2 = b^2 = -1$, $\epsilon T_{b,a} \epsilon 1 = b(a(\bar{1})a)b = 1$ and so $\epsilon T_{b,a} \epsilon \in \text{SO}(7)$. Therefore $(L_{b,a}, R_{b,a}, \epsilon T_{b,a} \epsilon) \in \tilde{B}_3$. Then by Lemma 2.3(1)(ii), $\epsilon T_{b,a} \epsilon = T_{b,a}$. Hence (3) follows.

(4) Put $G = \{(g_1, g_2, g_3) \in \tilde{B}_3 \mid g_1 1 = 1\}$. $G_2 \subset G$ follows from $g_1 1 = 1$ for all $g \in G_2$. Conversely, take $(g_1, g_2, g_3) \in G$. By virtue of the definition of \tilde{B}_3 , $g_2 = \epsilon g_1 \epsilon$, and so $g_2 1 = \epsilon g_1 \epsilon 1 = 1$. Thus we get $g_1 1 = g_2 1 = g_3 1 = 1$. Because of $g_i 1 = g_{i+1} 1 = 1$, applying Lemma 2.3(2), $g_{i+1} = \epsilon g_{i+2} \epsilon$ and $g_{i+2} = \epsilon g_i \epsilon$. Therefore $g_{i+1} = \epsilon g_{i+2} \epsilon = \epsilon(\epsilon g_i \epsilon) \epsilon = g_i$. Moving $i \in \{1, 2, 3\}$, $g_1 = g_2 = g_3$. Therefore $(g_1, g_2, g_3) \in G_2$, and so $G \subset G_2$. Hence $G = G_2$. Similarly, we have $G_2 = \{(g_1, g_2, g_3) \in \tilde{B}_3 \mid g_2 1 = 1\}$. \square

Lemma 2.5.

(1) $\tilde{B}_3/G_2 \simeq S^7$. Furthermore, \tilde{B}_3 is connected.

(2) $\tilde{D}_4/\tilde{B}_3 \simeq S^7$. Furthermore, \tilde{D}_4 is connected.

Proof. (1) We consider the action of \tilde{B}_3 on S^7 as $x \mapsto p_1(g_1, g_2, g_3)x = g_1 x$ for $x \in S^7$ and $(g_1, g_2, g_3) \in \tilde{B}_3$. Fix $x \in S^7$. Then x can be expressed by $x = \cos \theta + a \sin \theta$ for some $a \in S^6$ and $\theta \in \mathbb{R}$. At first, by Lemma 2.4(1), note that $S^6 = \text{Orb}_{G_2}(e_1)$. Thus there exists $h_1 \in G_2 \subset \tilde{B}_3$ such that $p_1(h_1)x = \cos \theta + e_1 \sin \theta$. At second, put $h_{e_1, e_2} = (L_{e_1, e_2}, R_{e_1, e_2}, T_{e_1, e_2})$. Since $e_i \in S^6$ and Lemma 2.4(3), $h_{e_1, e_2} \in \tilde{B}_3$. Then $p_1(h_{e_1, e_2})p_1(h_1)x = e_3 \cos \theta + e_2 \sin \theta \in S^6$. At third, since $S^6 = \text{Orb}_{G_2}(e_1)$, there exists $h_2 \in G_2$ such that $p_1(h_2)p_1(h_{e_1, e_2})p_1(h_1)x = e_1$. At last, put $h_{e_3, e_2} = (L_{e_3, e_2}, R_{e_3, e_2}, T_{e_3, e_2})$. Then it follows that $h_{e_3, e_2} \in \tilde{B}_3$ and $p_1(h_{e_3, e_2})p_1(h_2)p_1(h_{e_1, e_2})p_1(h_1)x = 1$. Hence \tilde{B}_3 acts transitively on S^7 . And by Lemma 2.4(4), $(\tilde{B}_3)_1 = G_2$. Therefore $\tilde{B}_3/G_2 \simeq S^7$. Since G_2 and S^7 are connected, \tilde{B}_3 is also connected. Hence (1) follows.

(2) We consider the action of \tilde{D}_4 on S^7 as $x \mapsto p_3(g_1, g_2, g_3)x = g_3 x$ for $x \in S^7$ and $(g_1, g_2, g_3) \in \tilde{D}_4$. Let $x \in S^7$. Then $\bar{x} \in S^7$. By Lemma 2.4(2), $(L_{\bar{x}}, R_{\bar{x}}, \epsilon T_{\bar{x}} \epsilon) \in \tilde{D}_4$. Therefore $(R_{\bar{x}}, \epsilon T_{\bar{x}} \epsilon, L_{\bar{x}}) \in \tilde{D}_4$ by Lemma 2.2. Then we get $p_3(R_{\bar{x}}, \epsilon T_{\bar{x}} \epsilon, L_{\bar{x}})x = \bar{x}x = 1$. Therefore \tilde{D}_4 acts transitively on S^7 . Hence $\tilde{D}_4/\tilde{B}_3 \simeq S^7$ follows from $(\tilde{D}_4)_1 = \tilde{B}_3$. Since \tilde{B}_3 is connected by (1) and S^7 is connected, \tilde{D}_4 is also connected. Hence (2) follows. \square

Proposition 2.6.

(1) (The principle of triality: [2], [9, (2.4.6)], cf. [40, Theorem 1.14.2])
The following sequence is exact:

$$1 \rightarrow \{(1, 1, 1), (\epsilon_i(1), \epsilon_i(2), \epsilon_i(3))\} \rightarrow \tilde{D}_4 \xrightarrow{p_i} \mathrm{SO}(8) \rightarrow 1.$$

(2) ([40, Theorem 1.15.2])
The following sequence is exact:

$$1 \rightarrow \{(1, 1, 1), (-1, -1, 1)\} \rightarrow \tilde{B}_3 \xrightarrow{q} \mathrm{SO}(7) \rightarrow 1.$$

By Lemma 2.5(2) and Proposition 2.6(1), \tilde{D}_4 is connected and a two-fold covering group of $\mathrm{SO}(8)$. And by Lemma 2.5(1) and Proposition 2.6(2), \tilde{B}_3 is connected and a two-fold covering group of $\mathrm{SO}(7)$. So let us denote

$$\mathrm{Spin}(8) := \tilde{D}_4, \quad \mathrm{Spin}(7) := \tilde{B}_3.$$

3. THE STABILIZER GROUPS OF SPIN GROUP TYPE.

Let $i \in \{1, 2, 3\}$, indices $i, i+1, i+2$ be counted modulo 3. The quadratic space $((\mathcal{J}^1)_{2E_i, -1}, Q_{E_i})$ is defined by

$$(\mathcal{J}^1)_{2E_i, -1} := \{X \in \mathcal{J}^1 \mid 2E_i \times X = -X\}, \quad Q_{E_i}(X) := -\mathrm{tr}(X^{\times 2}).$$

Since Lemma 1.3(1)(2) and direct calculation,

$$\begin{cases} (\mathcal{J}^1)_{2E_i, -1} = \{\xi(E_{i+1} - E_{i+2}) + F_i^1(x) \mid \xi \in \mathbb{R}, x \in \mathbf{O}\}, \\ Q_{E_i}(\xi(E_{i+1} - E_{i+2}) + F_i^1(x)) = \xi^2 + \epsilon_1(i)(x|x). \end{cases}$$

Then we see that $((\mathcal{J}^1)_{2E_1, -1}, Q_{E_1})$ is isomorphic to $(\mathbb{R}^{0,9}, Q_{0,9})$ and $((\mathcal{J}^1)_{2E_i, -1}, Q_{E_i})$ with $i = 2, 3$ is isomorphic to $(\mathbb{R}^{8,1}, Q_{8,1})$ and denote

$$\begin{aligned} \mathcal{S}^8 &:= \{X \in (\mathcal{J}^1)_{2E_1, -1} \mid Q_{E_1}(X) = 1\} \subset (\mathcal{J}^1)_{2E_1, -1}, \\ \mathcal{S}^{8,1} &:= \{X \in (\mathcal{J}^1)_{2E_3, -1} \mid Q_{E_3}(X) = 1\} \subset (\mathcal{J}^1)_{2E_3, -1}, \\ \mathcal{S}_+^{8,1} &:= \{X \in \mathcal{S}^{8,1} \mid (X|E_1) > 0\}, \\ \mathcal{S}_-^{8,1} &:= \{X \in \mathcal{S}^{8,1} \mid (X|E_1) < 0\}. \end{aligned}$$

The quadratic subspace $\mathcal{J}_{7,1}^1$ is defined by

$$\begin{aligned} \mathcal{J}_{7,1}^1 &:= \{X \in (\mathcal{J}^1)_{2E_3, -1} \mid (F_3^1(1)|X) = 0\} \\ &= \{\xi(E_1 - E_2) + F_3^1(x) \mid \xi \in \mathbb{R}, x \in \mathrm{Im}\mathbf{O}\}. \end{aligned}$$

Then $Q_{E_3}(\xi(E_1 - E_2) + F_3^1(x)) = \xi^2 - (x|x)$, and so $(\mathcal{J}_{7,1}^1, Q_{E_3})$ is isomorphic to $(\mathbb{R}^{7,1}, Q_{7,1})$ and denote

$$\mathcal{S}^{7,1} := \{X \in \mathcal{J}_{7,1}^1 \mid Q_{E_3}(X) = 1\}, \quad \mathcal{S}_+^{7,1} := \{X \in \mathcal{S}^{7,1} \mid (X|E_1) > 0\}.$$

Moreover $(F_i^1(\mathbf{O}), Q_{E_i})$ and $(F_i^1(\text{Im}\mathbf{O}), Q_{E_i})$ are the quadratic subspaces:

$$F_i^1(\mathbf{O}) = \{X \in (\mathcal{J}^1)_{2E_i, -1} \mid E_j \times X = 0, i \neq j\}$$

$$F_i^1(\text{Im}\mathbf{O}) = \{F_i^1(x) \mid x \in \text{Im}\mathbf{O}\} = \{X \in F_i^1(\mathbf{O}) \mid (F_i^1(1)|X) = 0\},$$

and $Q_{E_i}(F_i^1(x)) = \epsilon_1(i)(x|x)$. Thus $(F_i^1(\mathbf{O}), Q_{E_i})$ and $(F_i^1(\text{Im}\mathbf{O}), Q_{E_i})$ are isomorphic to $(\mathbb{R}^8, \epsilon_1(i)\mathfrak{n})$ and $(\mathbb{R}^7, \epsilon_1(i)\mathfrak{n})$, respectively. The stabilizer groups D_4 and B_3 of $F_{4(-20)}$ are defined as

$$D_4 := (F_{4(-20)})_{E_1, E_2, E_3}, \quad B_3 := (D_4)_{F_3^1(1)}$$

respectively and the homomorphisms as

$$\tilde{p}_i : (F_{4(-20)})_{E_i} \rightarrow \text{O}((\mathcal{J}^1)_{2E_i, -1}, Q_{E_i}), \quad \tilde{p}_i(g) := g|(\mathcal{J}^1)_{2E_i, -1},$$

$$\tilde{q} : (F_{4(-20)})_{F_3^1(1)} \rightarrow \text{O}(\mathcal{J}_{7,1}^1, Q_{E_3}), \quad \tilde{q}(g) := g|\mathcal{J}_{7,1}^1,$$

$$p_i : D_4 \rightarrow \text{O}(F_i^1(\mathbf{O}), Q_{E_i}), \quad p_i(g) := g|F_i^1(\mathbf{O}),$$

$$q : B_3 \rightarrow \text{O}(F_3^1(\text{Im}\mathbf{O}), Q_{E_3}), \quad q(g) := g|F_3^1(\text{Im}\mathbf{O})$$

respectively. Since $F_3^1(1)^{\times 2} = E_3$, $E = E_1 + E_2 + E_3$ and Proposition 1.5, we have:

Lemma 3.1.

- (1) $(F_{4(-20)})_{F_3^1(1)}$ is a subgroup of $(F_{4(-20)})_{E_3}$.
Especially, $(F_{4(-20)})_{F_3^1(1)} = (F_{4(-20)})_{E_3, F_3^1(1)}$.
- (2) $B_3 = (F_{4(-20)})_{E_1, F_3^1(1)} = (F_{4(-20)})_{E_2, F_3^1(1)}$.
- (3) The homomorphism \tilde{p}_i , \tilde{q} , p and q are well-defined.

The homomorphism $\varphi_0 : \text{Spin}(8) \rightarrow \text{GL}_{\mathbb{R}}(\mathcal{J}^1)$ is defined by

$$\varphi_0(g_1, g_2, g_3) \left(\sum_{i=1}^3 (\xi_i E_i + F_i^1(x_i)) \right) := \sum_{i=1}^3 (\xi_i E_i + F_i^1(g_i x_i)).$$

[22] (cf. [34]).

Lemma 3.2.

- (1) $\varphi_0 : \text{Spin}(8) \rightarrow D_4$ is an isomorphism onto D_4 .
- (2) $\varphi_0 : \text{Spin}(7) \rightarrow B_3$ is an isomorphism onto B_3 .

Proof. (1) It can be similarly proved as [40, Theorem 2.7.1].

(2) Fix $g \in B_3$. By (1), $g = \varphi_0(g_1, g_2, g_3)$ for some $(g_1, g_2, g_3) \in \text{Spin}(8)$. Then $F_3^1(1) = \varphi_0(g_1, g_2, g_3)F_3^1(1) = F_3^1(g_3 1)$. Therefore $g_3 1 = 1$, and so $(g_1, g_2, g_3) \in \text{Spin}(7)$. Therefore $\varphi_0 : \text{Spin}(7) \rightarrow B_3$ is onto. And by (1), φ_0 is a mono-morphism. Hence φ_0 is an isomorphism. \square

Hereafter we identify D_4 and B_3 with $\text{Spin}(8)$ and $\text{Spin}(7)$, respectively, via φ_0 . Let us denote $\mathfrak{f}_{4(-20)} = \text{Lie}(F_{4(-20)})$ and the subalgebra \mathfrak{d}_4 of $\mathfrak{f}_{4(-20)}$ as

$$\mathfrak{d}_4 := \{D \in \mathfrak{f}_{4(-20)} \mid DE_i = 0, i = 1, 2, 3\}.$$

[1] (cf. [9], [34]) and the Lie algebra $\tilde{\mathfrak{d}}_4$ as

$$\tilde{\mathfrak{d}}_4 := \{D \in \text{End}_{\mathbb{R}}(\mathbf{O}) \mid (Dx|y) + (x|Dy) = 0\}.$$

Lemma 3.3.

(1) (Principle of infinitesimal triality in $\tilde{\mathfrak{d}}_4$: [9], cf. [40, Lemma 1.3.6]). For all $D_1 \in \tilde{\mathfrak{d}}_4$, there exist $D_2, D_3 \in \tilde{\mathfrak{d}}_4$ such that

$$(D_1x)y + x(D_2y) = \epsilon D_3 \epsilon(xy) \quad \text{for all } x, y \in \mathbf{O}.$$

Also such D_2 and D_3 are uniquely determined for D_1 .

(2) (cf. [40, Lemma 1.3.7]) For $D_1, D_2, D_3 \in \tilde{\mathfrak{d}}_4$, the relation

$$(D_1x)y + x(D_2y) = \epsilon D_3 \epsilon(xy) \quad \text{for all } x, y \in \mathbf{O}$$

implies that for all $x, y \in \mathbf{O}$,

$$(D_2x)y + x(D_3y) = \epsilon D_1 \epsilon(xy), \quad (D_3x)y + x(D_1y) = \epsilon D_2 \epsilon(xy).$$

(3) (cf. Lemma 3.2(1)) \mathfrak{d}_4 is isomorphic to $\tilde{\mathfrak{d}}_4$ under the correspondence $D_1 \mapsto d\varphi_0(D_1, D_2, D_3)$ given by

$$d\varphi_0(D_1, D_2, D_3) \left(\sum_{i=1}^3 (\xi_i E_i + F_i^1(x_i)) \right) := \sum_{i=1}^3 F_i^1(D_i x_i).$$

where D_2 and D_3 are elements of $\tilde{\mathfrak{d}}_4$ which are determined by D_1 from the infinitesimal triality:

$$(D_1x)y + x(D_2y) = \epsilon D_3 \epsilon(xy) \quad \text{for all } x, y \in \mathbf{O}.$$

Lemma 3.4. Let $g = (g_1, g_2, g_3) \in \text{Spin}(7)$ and $p, x \in \mathbf{O}$.

$$\begin{cases} \text{(i)} & \varphi_0(g)(-E_1 + E_2) = -E_1 + E_2, \\ \text{(ii)} & \varphi_0(g)P^- = P^-, \quad \text{(iii)} & \varphi_0(g)F_3^1(p) = F_3^1(g_3p), \\ \text{(iv)} & \varphi_0(g)E_3 = E_3, \quad \text{(v)} & \varphi_0(g)E = E, \\ \text{(vi)} & \varphi_0(g)Q^+(x) = Q^+(g_1x), \\ \text{(vii)} & \varphi_0(g)Q^-(x) = Q^-(g_1x). \end{cases}$$

Proof. The first five equations follow from the definition of φ_0 and $\text{Spin}(7)$. Since $g_2 = \epsilon g_1 \epsilon$, $\varphi_0(g)F_1^1(\bar{x}) = F_2^1(\epsilon g_1 \epsilon \bar{x}) = F_2^1(\overline{g_1 x})$. Thus $\varphi_0(g)(F_1^1(x) + \pm F_2^1(\bar{x})) = F_1^1(g_1 x) \pm F_2^1(\overline{g_1 x})$. Hence the last two equations follow. \square

Proposition 3.5. Let $Y = \text{diag}(r_1, r_2, r_3) \in \mathcal{J}^1$ where $(r_1 - r_2)(r_2 - r_3)(r_3 - r_1) \neq 0$. Then $(F_{4(-20)})_Y = D_4$.

Proof. Let $g \in D_4$. Then $gY = g(\sum_{i=1}^3 r_i E_i) = \sum_{i=1}^3 r_i E_i = Y$. Therefore $g \in (F_{4(-20)})_Y$, and so $D_4 \subset (F_{4(-20)})_Y$.

Conversely, take $g \in (F_{4(-20)})_Y$. Then $\varphi_Y(r_i)^{\times 2} = (r_{i+1} - r_i)(r_{i+2} - r_i)E_i$ and $\text{tr}(\varphi_Y(r_i)^{\times 2}) = (r_{i+1} - r_i)(r_{i+2} - r_i) \neq 0$ where indices $i, i+1, i+2$ are counted modulo 3. Therefore E_{Y, r_i} is well-defined and $E_{Y, r_i} =$

E_i . By Proposition 1.9(2), $gE_i = gE_{Y,r_i} = E_{gY,r_i} = E_{Y,r_i} = E_i$. Thus $g \in D_4$ and so $(F_{4(-20)})_Y \subset D_4$. Hence $(F_{4(-20)})_Y = D_4$. \square

For $a \in \mathbf{O}$, let us denote

$$\begin{aligned} A_1^1(a) &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, \quad A_2^1(a) := \begin{pmatrix} 0 & 0 & -\sqrt{-1}\bar{a} \\ 0 & 0 & 0 \\ \sqrt{-1}a & 0 & 0 \end{pmatrix}, \\ A_3^1(a) &:= \begin{pmatrix} 0 & \sqrt{-1}a & 0 \\ -\sqrt{-1}\bar{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

For $i \in \{1, 2, 3\}$, $\tilde{A}_i^1(a) \in \text{End}_{\mathbb{R}}(\mathcal{J}^1)$ and the subspace $\tilde{\mathbf{u}}_i^1$ of $\text{End}_{\mathbb{R}}(\mathcal{J}^1)$ are defined as

$$(3.1) \quad \begin{aligned} \tilde{A}_i^1(a)X &:= A_i(a)X - XA_i(a) \quad \text{for } X \in \mathcal{J}^1, \\ \tilde{\mathbf{u}}_i^1 &:= \{\tilde{A}_i^1(a) \mid a \in \mathbf{O}\} \end{aligned}$$

respectively [9] (cf. [22], [34]). By direct calculation, we have:

Lemma 3.6.

- (1) $\mathfrak{f}_{4(-20)} = \mathfrak{d}_4 \oplus \tilde{\mathbf{u}}_1^1 \oplus \tilde{\mathbf{u}}_2^1 \oplus \tilde{\mathbf{u}}_3^1$.
- (2) The operation of $\tilde{A}_i^1(a)$ on \mathcal{J}^1 is given by
 - $\left\{ \begin{aligned} \text{(i)} \quad & \tilde{A}_i^1(a)E_i = 0, \quad \text{(ii)} \quad \tilde{A}_i^1(a)E_{i+1} = -F_i^1(a), \quad \text{(iii)} \quad \tilde{A}_i^1(a)E_{i+2} = F_i^1(a), \\ \text{(iv)} \quad & \tilde{A}_i^1(a)F_i^1(x) = \epsilon_1(i)2(a|x)(E_{i+1} - E_{i+2}), \\ \text{(v)} \quad & \tilde{A}_i^1(a)F_{i+1}^1(x) = -\epsilon_1(i+2)F_{i+2}^1(\bar{a}x), \\ \text{(vi)} \quad & \tilde{A}_i^1(a)F_{i+2}^1(x) = \epsilon_1(i+1)F_{i+1}^1(\bar{x}a). \end{aligned} \right.$

Lemma 3.7. Let $t \in \mathbb{R}$, $a \in S^7$, $\xi, \eta \in \mathbb{R}^3$, $x, y \in \mathbf{O}^3$ and indices be counted modulo 3.

- (1) Let $h^1(\eta; y) \in \mathcal{J}^1$ be

$$\begin{cases} \eta_1 &= \xi_1, \\ \eta_2 &= \frac{\xi_2 + \xi_3}{2} + \frac{\xi_2 - \xi_3}{2} \cos 2t + (a|x_1) \sin 2t, \\ \eta_3 &= \frac{\xi_2 + \xi_3}{2} - \frac{\xi_2 - \xi_3}{2} \cos 2t - (a|x_1) \sin 2t, \\ y_1 &= x_1 - \frac{\xi_2 - \xi_3}{2} a \sin 2t - 2(a|x_1)a \sin^2 t, \\ y_2 &= x_2 \cos t - \bar{x}_3 a \sin t, \\ y_3 &= x_3 \cos t + \bar{a}x_2 \sin t. \end{cases}$$

Then $h^1(\eta; y) = \exp(t\tilde{A}_1^1(a))h^1(\xi; x)$ and $\exp(t\tilde{A}_1^1(a)) \in (F_{4(-20)})_{E_1}^0$.

- (2) Assume that $i \in \{2, 3\}$. Let $h^1(\eta; y) \in \mathcal{J}^1$ be

$$\begin{cases} \eta_i &= \xi_i, \\ \eta_{i+1} &= \frac{\xi_{i+1} + \xi_{i+2}}{2} + \frac{\xi_{i+1} - \xi_{i+2}}{2} \cosh 2t - (a|x_i) \sinh 2t, \\ \eta_{i+2} &= \frac{\xi_{i+1} + \xi_{i+2}}{2} - \frac{\xi_{i+1} - \xi_{i+2}}{2} \cosh 2t + (a|x_i) \sinh 2t, \\ y_i &= x_i - \frac{\xi_{i+1} - \xi_{i+2}}{2} a \sinh 2t + 2(a|x_i)a \sinh^2 t, \\ y_{i+1} &= x_{i+1} \cosh t + \bar{x}_{i+2} a \sinh t, \\ y_{i+2} &= x_{i+2} \cosh t + \bar{a}x_i \sinh t. \end{cases}$$

Then $h^1(\eta; y) = \exp(t\tilde{A}_i^1(a))h^1(\xi; x)$ and $\exp(t\tilde{A}_i^1(a)) \in (F_{4(-20)})_{E_i}^0$.

(3) Let $i \in \{1, 2, 3\}$. If $a \in S^6$, then $\exp(t\tilde{A}_i^1(a)) \in (F_{4(-20)})_{F_i^1(1)}^0$.

Proof. (1)(2) Let $j \in \{1, 2, 3\}$ and $F : \mathbb{R} \times \mathcal{J}^1 \rightarrow \mathcal{J}^1$ be the map defined by $F(t, h^1(\xi; x)) = h^1(\eta; y)$. By direct calculation, $\frac{d}{dt}F(t, h^1(\xi; x)) = \tilde{A}_j^1(a)F(t, h^1(\xi; x))$ and $F(0, h^1(\xi; x)) = h^1(\xi; x)$. Using the uniqueness of solutions, we get $F(t, h^1(\xi; x)) = \exp(t\tilde{A}_j^1(a))h^1(\xi; x)$. Since $\eta_j = \xi_j$, $\exp(t\tilde{A}_j^1(a))E_j = E_j$. Hence the assertions follows from $\exp(t\tilde{A}_j^1(a)) \in (F_{4(-20)})^0$.

(3) Since (1), (2) and $a|1 = 0$, $\exp(t\tilde{A}_i^1(a))F_i^1(1) = F_i^1(1)$. Hence (3) follows from $\exp(t\tilde{A}_i^1(a)) \in (F_{4(-20)})^0$. \square

Lemma 3.8.

(1) ([24, Proposition 2.8(1)]) $S^8 = \text{Orb}_{(F_{4(-20)})_{E_1}}(E_2 - E_3)$.

(2) ([24, Proposition 2.8(3)])

(i) $\mathcal{S}_+^{8,1} = \text{Orb}_{(F_{4(-20)})_{E_3}}(E_1 - E_2) = \text{Orb}_{(F_{4(-20)})_{E_3}^0}(E_1 - E_2)$,

(ii) $\mathcal{S}_-^{8,1} = \text{Orb}_{(F_{4(-20)})_{E_3}}(-E_1 + E_2) = \text{Orb}_{(F_{4(-20)})_{E_3}^0}(-E_1 + E_2)$.

Furthermore, $\mathcal{S}_+^{8,1}$ is connected.

Lemma 3.9.

(1) $(F_{4(-20)})_{E_1}/D_4 \simeq S^8$. Furthermore, $(F_{4(-20)})_{E_1}$ is connected.

(2) $(F_{4(-20)})_{E_3}/D_4 \simeq \mathcal{S}_+^{8,1}$. Furthermore, $(F_{4(-20)})_{E_3}$ is connected.

Proof. (1) Because of $E_2 = \frac{1}{2}(E - E_1 + (E_2 - E_3))$ and $E_3 = \frac{1}{2}(E - E_1 - (E_2 - E_3))$, $D_4 = (F_{4(-20)})_{E_1, E_2 - E_3}$. Therefore $(F_{4(-20)})_{E_1}/D_4 \simeq S^8$ follows from Lemma 3.8(1). By Lemma 3.2(1), D_4 is connected, and obviously S^8 is connected. Hence $(F_{4(-20)})_{E_1}$ is also connected.

(2) By Lemma 3.8(2)(i), $\mathcal{S}_+^{8,1} = \text{Orb}_{(F_{4(-20)})_{E_3}}(E_1 - E_2)$. Similarly, $D_4 = (F_{4(-20)})_{E_3, E_1 - E_2}$. Therefore $(F_{4(-20)})_{E_3}/D_4 \simeq \mathcal{S}_+^{8,1}$ follows from Lemma 3.8(2)(i). By Lemma 3.2(1), D_4 is connected, and since $\mathcal{S}_+^{8,1}$ is a orbit of one element under the action of a connected group $(F_{4(-20)})_{E_3}^0$ by Lemma 3.8(2)(i), $\mathcal{S}_+^{8,1}$ is connected. Hence $(F_{4(-20)})_{E_3}$ is also connected. \square

For $i \in \{1, 2, 3\}$, the element $\sigma_i \in F_{4(-20)}$ is defined by

$$(3.2) \quad \sigma_i \left(\sum_{j=1}^3 (\xi_j E_j + F_j^1(x_j)) \right) := \sum_{j=1}^3 (\xi_j E_j + \epsilon_i(j) F_j^1(x_j))$$

[35] (cf. [36]). In fact, because of $\det(\sigma_i X) = \det(X)$ and $\sigma_i E = E$, $\sigma_i \in F_{4(-20)}$. Moreover, $\sigma_i^2 = 1$.

Proposition 3.10.

(1) *The following sequence is exact:*

$$1 \rightarrow \{1, \sigma_i\} \rightarrow D_4 \xrightarrow{p_i} \mathrm{SO}(F_i^1(\mathbf{O}), Q_{E_i}) \rightarrow 1.$$

(2) ([36, Theorem 2.4.4(1)]) *The following sequence is exact:*

$$1 \rightarrow \{1, \sigma_1\} \rightarrow (F_{4(-20)})_{E_1} \xrightarrow{\tilde{p}_1} \mathrm{SO}((\mathcal{J}^1)_{2E_1, -1}, Q_{E_1}) \rightarrow 1.$$

(3) ([36, Theorem 2.4.4(3)]) *The following sequence is exact:*

$$1 \rightarrow \{1, \sigma_3\} \rightarrow (F_{4(-20)})_{E_3} \xrightarrow{\tilde{p}_3} \mathrm{O}^0((\mathcal{J}^1)_{2E_3, -1}, Q_{E_3}) \rightarrow 1.$$

Proof. (1) It follows from Lemma 3.2(1) and Proposition 2.6(1).

(2) Since $(F_{4(-20)})_{E_1}$ is connected by Lemma 3.9(1), $\tilde{p}_1((F_{4(-20)})_{E_1}) \subset \mathrm{SO}((\mathcal{J}^1)_{2E_1, -1}, Q_{E_1})$. Then the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \rightarrow & D_4 & \rightarrow & (F_{4(-20)})_{E_1} & \rightarrow & S^8 \rightarrow * \\ & & \downarrow p_1 & & \downarrow \tilde{p}_1 & & \parallel \\ 1 & \rightarrow & \mathrm{SO}(F_1^1(\mathbf{O}), Q_{E_1}) & \rightarrow & \mathrm{SO}((\mathcal{J}^1)_{2E_1, -1}, Q_{E_1}) & \rightarrow & S^8 \rightarrow *. \end{array}$$

Using five lemma, it follows that \tilde{p}_1 is onto from p_1 is onto, and $\mathrm{Ker}(\tilde{p}_1) = \mathrm{Ker}(p_1) = \{1, \sigma_1\}$. Hence (2) follows.

(3) Since $(F_{4(-20)})_{E_3}$ is connected by Lemma 3.9(2), $\tilde{p}_3((F_{4(-20)})_{E_3}) \subset \mathrm{O}^0((\mathcal{J}^1)_{2E_3, -1}, Q_{E_3})$. Then the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \rightarrow & D_4 & \rightarrow & (F_{4(-20)})_{E_3} & \rightarrow & \mathcal{S}_+^{8,1} \rightarrow * \\ & & \downarrow p_3 & & \downarrow \tilde{p}_3 & & \parallel \\ 1 & \rightarrow & \mathrm{SO}(F_3^1(\mathbf{O}), Q_{E_3}) & \rightarrow & \mathrm{O}^0((\mathcal{J}^1)_{2E_3, -1}, Q_{E_3}) & \rightarrow & \mathcal{S}_+^{8,1} \rightarrow *. \end{array}$$

Similarly, using five lemma, (3) follows. \square

By Lemma 3.9(1) and Proposition 3.10(2), $(F_{4(-20)})_{E_1}$ is connected and a two-fold covering group of $\mathrm{SO}((\mathcal{J}^1)_{2E_1, -1}, Q_{E_1})$. Moreover by Lemma 3.9(2) and Proposition 3.10(3), $(F_{4(-20)})_{E_3}$ is connected and a two-fold covering group of $\mathrm{O}^0((\mathcal{J}^1)_{2E_3, -1}, Q_{E_3})$. So let us denote

$$\mathrm{Spin}(9) := (F_{4(-20)})_{E_1}, \quad \mathrm{Spin}^0(8, 1) := (F_{4(-20)})_{E_3}.$$

Proposition 3.11.

(1) *Let $Y = (r_1 - r_2)E_1 + r_2E \in \mathcal{J}^1$ where $r_1 \neq r_2$. Then*

$$(F_{4(-20)})_Y = \mathrm{Spin}(9).$$

(2) *Let $Y' = (r_1 - r_2)E_3 + r_2E \in \mathcal{J}^1$ where $r_1 \neq r_2$. Then*

$$(F_{4(-20)})_{Y'} = \mathrm{Spin}^0(8, 1).$$

Proof. Since the element E is invariant under the $F_{4(-20)}$ -action, we obtain $(F_{4(-20)})_Y = (F_{4(-20)})_{E_1}$ and $(F_{4(-20)})_{Y'} = (F_{4(-20)})_{E_3}$. \square

Lemma 3.12.

$$(1) \mathcal{S}_+^{7,1} = \text{Orb}_{(F_4(-20))_{F_3^1(1)}^0} (E_1 - E_2) = \text{Orb}_{(F_4(-20))_{F_3^1(1)}} (E_1 - E_2).$$

Furthermore, $\mathcal{S}_+^{7,1}$ is connected.

$$(2) (F_4(-20))_{F_3^1(1)}/B_3 \simeq \mathcal{S}_+^{7,1}. \text{ Furthermore, } (F_4(-20))_{F_3^1(1)} \text{ is connected.}$$

Proof. (1) Note $\mathcal{S}_+^{7,1} = \mathcal{S}_+^{8,1} \cap \mathcal{S}_+^{7,1}$. Fix $X \in \mathcal{S}_+^{7,1}$ and $g \in (F_4(-20))_{F_3^1(1)}$. By Lemmas 3.1(1) and 3.8(2)(i), $gX \in \mathcal{S}_+^{8,1}$ and $0 = (F_3^1(1)|X) = (gF_3^1(1)|gX) = (F_3^1(1)|gX)$. Therefore $gX \in \mathcal{S}_+^{7,1}$, and so $(F_4(-20))_{F_3^1(1)}$ acts on $\mathcal{S}_+^{7,1}$. Especially, $(F_4(-20))_{F_3^1(1)}^0$ acts on $\mathcal{S}_+^{7,1}$.

Next, we will show transitivity. Fix $X \in \mathcal{S}_+^{7,1}$. X is expressed by $X = \xi(E_1 - E_2) + F_3^1(x)$ where $\xi > 0$, $x \in \text{Im } \mathbf{O}$ and $\xi^2 - \mathbf{n}(x) = 1$. By Lemma 2.4(1), $gx = \sqrt{\mathbf{n}(x)}e_1$ for some $g \in G_2$. So $\varphi_0(g, g, g) \in G_2 \subset B_3 \subset (F_4(-20))_{F_3^1(1)}^0$ and

$$\varphi_0(g, g, g)X = \xi(E_1 - E_2) + F_3^1(\sqrt{\mathbf{n}(x)}e_1).$$

By Lemma 3.7(3), $\exp(t\tilde{A}_3^1(e_1)) \in (F_4(-20))_{F_3^1(1)}^0$. Since $\xi^2 - \mathbf{n}(x) = 1$, Lemma 3.7(2) and direct calculation,

$$\exp\left(\frac{1}{4} \log\left(\frac{\xi + \sqrt{\mathbf{n}(x)}}{\xi - \sqrt{\mathbf{n}(x)}}\right)\tilde{A}_3^1(e_1)\right)\varphi_0(g, g, g)X = E_1 - E_2.$$

Therefore we obtain

$$\mathcal{S}_+^{7,1} = \text{Orb}_{(F_4(-20))_{E_3}^0} (E_1 - E_2) = \text{Orb}_{(F_4(-20))_{E_3}} (E_1 - E_2).$$

Since $\mathcal{S}_+^{7,1}$ is an orbit of one element under the action of a connected group $(F_4(-20))_{E_3}^0$, $\mathcal{S}_+^{7,1}$ is connected. Hence (1) follows.

(2) Since $(F_4(-20))_{E_1-E_2, E_3} = (F_4(-20))_{E, E_1-E_2, E_3} = (F_4(-20))_{E_1, E_2, E_3}$ and Lemma 3.1(1)(2), $(F_4(-20))_{F_3^1(1), E_1-E_2} = (F_4(-20))_{E_1, E_3, F_3^1(1)} = B_3$. Hence $(F_4(-20))_{F_3^1(1)}/B_3 \simeq \mathcal{S}_+^{7,1}$ follows from (1). By Lemma 3.2(2) and (1), B_3 and $\mathcal{S}_+^{7,1}$ are connected. Therefore $(F_4(-20))_{F_3^1(1)}$ is also connected. \square

Proposition 3.13.

(1) The following sequence is exact:

$$1 \rightarrow \{1, \sigma_3\} \rightarrow B_3 \xrightarrow{q} \text{SO}(F_3^1(\text{Im } \mathbf{O}), Q_{E_3}) \rightarrow 1.$$

(2) The following sequence is exact:

$$1 \rightarrow \{1, \sigma_3\} \rightarrow (F_4(-20))_{F_3^1(1)} \xrightarrow{\tilde{q}} \text{O}^0(\mathcal{J}_{7,1}^1, Q_{E_3}) \rightarrow 1.$$

Proof. (1) It follows from Lemma 3.2(2) and Proposition 2.6(2).

(2) Since $(F_{4(-20)})_{F_3^1(1)}$ is connected by Lemma 3.12(2), we obtain $\tilde{q}((F_{4(-20)})_{F_3^1(1)}) \subset O^0(\mathcal{J}_{7,1}^1, Q_{E_3})$ and following commutative diagram:

$$\begin{array}{ccccccc} 1 & \rightarrow & B_3 & \rightarrow & (F_{4(-20)})_{F_3^1(1)} & \rightarrow & \mathcal{S}_+^{7,1} \rightarrow * \\ & & \downarrow q & & \downarrow \tilde{q} & & \parallel \\ 1 & \rightarrow & \mathrm{SO}(F_3^1(\mathrm{Im} \mathbf{O}), Q_{E_3}) & \rightarrow & O^0(\mathcal{J}_{7,1}^1, Q_{E_3}) & \rightarrow & \mathcal{S}_+^{7,1} \rightarrow *. \end{array}$$

Using five lemma, it follows that \tilde{q} is onto from q is onto, and $\mathrm{Ker}(\tilde{q}) = \mathrm{Ker}(q) = \{1, \sigma_3\}$. Hence the assertion follows. \square

By Lemma 3.12(2) and Proposition 3.13(2), $(F_{4(-20)})_{F_3^1(1)}$ is connected and a two-fold covering group of $O^0(\mathcal{J}_{7,1}^1, Q_{E_3})$. So let us denote

$$\mathrm{Spin}^0(7, 1) := (F_{4(-20)})_{F_3^1(1)}.$$

Proposition 3.14. *Let $Y = rE_3 + p(E - E_3) + qF_3^1(1) \in \mathcal{J}^1$ where $q \neq 0$. Then*

$$(F_{4(-20)})_Y = \mathrm{Spin}^0(7, 1).$$

Proof. Let $g \in \mathrm{Spin}^0(7, 1)$. By Lemma 3.1(1), $gY = g(rE_3 + p(E - E_3) + qF_3^1(1)) = rE_3 + p(E - E_3) + qF_3^1(1) = Y$. Therefore $g \in (F_{4(-20)})_Y$, and so $\mathrm{Spin}^0(7, 1) \subset (F_{4(-20)})_Y$.

Conversely, take $g \in (F_{4(-20)})_Y$. Then $\varphi_Y(r)^{\times 2} = ((p - r)^2 + q^2)E_3$ and $\mathrm{tr}(\varphi_Y(r)^{\times 2}) = (p - r)^2 + q^2 \neq 0$, and so $E_{Y,r} \in \mathcal{J}^1$ is well-defined and $E_{Y,r} = E_3$. Then by Proposition 1.9(2), $gE_3 = gE_{Y,r} = E_{gY,r} = E_{Y,r} = E_3$. Therefore $gF_3^1(1) = g(\frac{1}{q}(Y - (r - p)E_3 - p(E - E_3))) = F_3^1(1)$ and so $g \in \mathrm{Spin}^0(7, 1)$. Therefore $(F_{4(-20)})_Y \subset \mathrm{Spin}^0(7, 1)$. Hence $(F_{4(-20)})_Y = \mathrm{Spin}^0(7, 1)$. \square

For $i \in \{1, 2, 3\}$, the involutive automorphism $\tilde{\sigma}_i$ of $F_{4(-20)}$ is defined by $\tilde{\sigma}_i(g) := \sigma_i g \sigma_i$ for $g \in F_{4(-20)}$ and the subgroup K of $F_{4(-20)}$ as

$$K := (F_{4(-20)})^{\tilde{\sigma}_1} = \{g \in F_{4(-20)} \mid \sigma_1 g = g \sigma_1\}$$

[35] (cf. [36]). Let indices be counted modulo 3 and denote

$$\mathcal{J}_{\sigma_i}^1 := \{X \in \mathcal{J}^1 \mid \sigma_i X = X\} = \{(\sum_{j=1}^3 \xi_j E_j) + F_i^1(x) \mid \xi_j \in \mathbb{R}, x \in \mathbf{O}\},$$

$$\mathcal{J}_{-\sigma_i}^1 := \{X \in \mathcal{J}^1 \mid \sigma_i X = -X\} = \{\sum_{j=1}^2 F_{i+1}^1(x_{i+j}) \mid x_{i+j} \in \mathbf{O}\}.$$

Lemma 3.15. *Let indices $i, i + 1, i + 2$ be counted modulo 3.*

- (1) $\begin{cases} \mathcal{J}_{\sigma_i}^1 &= \{X \in \mathcal{J}^1 \mid 4E_i \times (E_i \times X) = X\} \oplus \mathbb{R}E_i, \\ \mathcal{J}_{-\sigma_i}^1 &= \{X \in \mathcal{J}^1 \mid E_i \times X = 0, (E_i \mid X) = 0\}. \end{cases}$
- (2) $(F_{4(-20)})^{\tilde{\sigma}_i} \mathcal{J}_{\pm \sigma_i}^1 = \mathcal{J}_{\pm \sigma_i}^1$ (resp).
- (3) Let $g \in (F_{4(-20)})^{\tilde{\sigma}_i}$. Then

$$gE_i = E_i + \xi_{i+1}E_{i+1} + \xi_{i+2}E_{i+2} + F_i^1(x)$$

for some $\xi_{i+1}, \xi_{i+2} \in \mathbb{R}$ and $x \in \mathbf{O}$.

Proof. (1) It follows from Lemma 1.3(1)(2) and direct calculation.

(2) It follows from $\sigma_i g = g \sigma_i$ for all $g \in (F_{4(-20)})^{\tilde{\sigma}_i}$.

(3) Let indices be counted modulo 3. Because of $F_{i+1}^1(1), F_{i+2}^1(1) \in \mathcal{J}_{-\sigma_i}^1$ and (2), $gF_{i+1}^1(1) = F_{i+1}^1(x_{i+1}) + F_{i+2}^1(x_{i+2})$ and $gF_{i+1}^1(1) = F_{i+1}^1(y_{i+1}) + F_{i+2}^1(y_{i+2})$ for some $x_{i+j}, y_{i+j} \in \mathbf{O}$. Then we get $gE_{i+1} = -\frac{(gF_{i+1}^1(1))^{\times 2}}{\epsilon_1(i+1)} = -\frac{(F_{i+1}^1(x_{i+1}) + F_{i+2}^1(x_{i+2}))^{\times 2}}{\epsilon_1(i+1)} = (\sum_{k=1}^2 \xi_{i+k} E_{i+k}) + F_i^1(u)$ and $gE_{i+2} = -\frac{(gF_{i+2}^1(1))^{\times 2}}{\epsilon_1(i+2)} = -\frac{(F_{i+1}^1(y_{i+1}) + F_{i+2}^1(y_{i+2}))^{\times 2}}{\epsilon_1(i+2)} = (\sum_{k=1}^2 \eta_{i+k} E_{i+k}) + F_i^1(v)$ for some $\xi_{i+k}, \eta_{i+k} \in \mathbb{R}$ and $u, v \in \mathbf{O}$. Thus $gE_i = g(E - E_{i+1} - E_{i+2}) = E - \sum_{k=1}^2 (\xi_{i+k} + \eta_{i+k}) E_{i+k} - F_i^1(u + v)$. Hence (3) follows. \square

Proposition 3.16. ([36, Theorem 2.4.4]) *Let $i \in \{1, 2, 3\}$.*

(1) $(F_{4(-20)})^{\tilde{\sigma}_i} = (F_{4(-20)})_{E_i}$.

(2) $K = (F_{4(-20)})_{E_1} = \text{Spin}(9)$.

(3) $(F_{4(-20)})^{\tilde{\sigma}_2} = (F_{4(-20)})_{E_2} \cong \text{Spin}^0(8, 1)$.

(4) K and $(F_{4(-20)})^{\tilde{\sigma}_2}$ are analytic subgroups of $F_{4(-20)}$.

Proof. (1) Let indices be counted modulo 3. For all $X \in \mathcal{J}^1$, X can be expressed by $X = X_{\sigma_i} + X_{-\sigma_i}$ for some $X_{\sigma_i} \in \mathcal{J}_{\sigma_i}^1$ and $X_{-\sigma_i} \in \mathcal{J}_{-\sigma_i}^1$. By Lemma 3.15(2), $g\sigma_i X = gX_{\sigma_i} - gX_{-\sigma_i} = \sigma_i gX$, and so $g\sigma_i = \sigma_i g$. Therefore $g \in (F_{4(-20)})^{\tilde{\sigma}_i}$ and so $(F_{4(-20)})_{E_i} \subset (F_{4(-20)})^{\tilde{\sigma}_i}$.

Conversely, take $g \in (F_{4(-20)})^{\tilde{\sigma}_i}$. By Lemma 3.15(3), $gE_i = E_i + \xi_{i+1}E_{i+1} + \xi_{i+2}E_{i+2} + F_i^1(x)$ for some $\xi_{i+1}, \xi_{i+2} \in \mathbb{R}$ and $x \in \mathbf{O}$. By Proposition 1.6, $gE_i \in \text{Orb}_{F_{4(-20)}}(E_i) \subset \mathcal{H}$. Since $(gE_i)^{\times 2} = 0$, we get $((gE_i)^{\times 2})_{E_{i+j}} = 0$ and $((gE_i)^{\times 2})_{E_i} = 0$ where $j \in \{1, 2\}$. Then by Lemma 1.3(2), $0 = ((gE_i)^{\times 2})_{E_{i+j}} = \xi_{i+j}$ and $0 = ((gE_i)^{\times 2})_{E_i} = \xi_{i+1}\xi_{i+2} - \epsilon_1(i)(x|x)$. Therefore $\xi_{i+1} = \xi_{i+2} = 0$ and $x = 0$, and so $gE_i = E_i$. Thus $g \in (F_{4(-20)})_{E_i}$ and so $(F_{4(-20)})^{\tilde{\sigma}_i} \subset (F_{4(-20)})_{E_i}$. Hence (1) follows.

(2) It follows from (1).

(3) Since $E_2 \in \text{Orb}_{F_{4(-20)}}(E_3)$, $(F_{4(-20)})_{E_2} \cong (F_{4(-20)})_{E_3} = \text{Spin}^0(8, 1)$. Thus (3) follows from (1).

(4) It follows from (2) and (3). \square

4. THE CONSTRUCTION OF NILPOTENT SUBGROUP.

Let \mathfrak{g} be a Lie algebra over $F = \mathbb{R}$ or \mathbb{C} . Let us denote tr as the trace of a vector space endomorphism, and for all $X \in \mathfrak{g}$, $\text{ad}(X) \in \text{End}_F(\mathfrak{g})$ as $\text{ad}(X) = [X, Y]$ for $Y \in \mathfrak{g}$. The *Killing form* B of \mathfrak{g} is defined by the bilinear form $B(X, Y) := \text{tr}(\text{ad}(X)\text{ad}(Y))$ for $X, Y \in \mathfrak{g}$. An involutive automorphism θ of a semisimple Lie algebra \mathfrak{g} is called a *Cartan involution* if the bilinear form $B(X, \theta Y)$ is negative definite. Then θ has two eigenvalues: 1 and -1 , so that put \mathfrak{k} and \mathfrak{p} be the

eigenspaces of θ , respectively. Then the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and call the *Cartan decomposition*.

The differential $d\tilde{\sigma}_i \in \text{Aut}_{\mathbb{R}}(\mathfrak{f}_{4(-20)})$ of the involutive automorphism $\tilde{\sigma}_i$ is written by same letter $\tilde{\sigma}_i : \tilde{\sigma}_i\phi = \sigma_i\phi\sigma_i$ for $\phi \in \mathfrak{f}_{4(-20)}$.

Lemma 4.1. *Let indices $i, i+1, i+2$ be counted modulo 3, $D \in \mathfrak{d}_4$ and $a, x, y \in \mathbf{O}$.*

(1) *The following equations hold.*

$$\begin{cases} \text{(i)} & \tilde{\sigma}_i D = D, & \text{(ii)} & \tilde{\sigma}_i \tilde{A}_i^1(a) = \tilde{A}_i^1(a), \\ \text{(iii)} & \tilde{\sigma}_i \tilde{A}_j^1(a) = -\tilde{A}_j^1(a), & & \text{for } j = i+1, i+2. \end{cases}$$

(2) *The following equations hold.*

$$\begin{cases} \text{(i)} & [D, \tilde{A}_i^1(x)] = \tilde{A}_i^1(D_i x), & \text{(ii)} & [\tilde{A}_i^1(x), \tilde{A}_i^1(y)] \in \mathfrak{d}_4, \\ \text{(iii)} & [\tilde{A}_i^1(x), \tilde{A}_{i+1}^1(y)] = \epsilon_1(i+2)\tilde{A}_{i+2}^1(\overline{xy}) \end{cases}$$

where $D = d\varphi_0(D_1, D_2, D_3) \in \mathfrak{d}_4$ (cf. Lemma 3.3(3)).

Proof. Let $X \in \{E_i, F_i^1(x) \mid x \in \mathbf{O}, i = 1, 2, 3\}$. Since Lemma 3.6(2) and direct calculation, $\sigma_i D \sigma_i X = DX$, $\sigma_i \tilde{A}_i(a) \sigma_i X = \tilde{A}_i(a)X$ and $\sigma_i \tilde{A}_j(a) \sigma_i X = -\tilde{A}_j(a)X$. Therefore (1) follows. Similarly, we obtain $[D, \tilde{A}_i^1(x)]X = \tilde{A}_i^1(D_i x)X$, $[\tilde{A}_i^1(x), \tilde{A}_{i+1}^1(y)]X = \epsilon_1(i+2)\tilde{A}_{i+2}^1(\overline{xy})X$ and $[\tilde{A}_i^1(x), \tilde{A}_i^1(y)]E_k = 0$ for all $k \in \{1, 2, 3\}$. Hence (2) follows. \square

Lemma 4.2.

(1) ([40, Theorem 2.5.3]) *The Killing form B of $\mathfrak{f}_{4(-20)}$ is given by*

$$B(\phi_1, \phi_2) = 3\text{tr}(\phi_1\phi_2) \quad \text{for all } \phi_i \in \mathfrak{f}_{4(-20)}.$$

(2) *Let $\phi = d\varphi(D_1, D_2, D_3) + \sum_{i=1}^3 \tilde{A}_i^1(a_i)$ where $d\varphi(D_1, D_2, D_3) \in \mathfrak{d}_4$ and $a_i \in \mathbf{O}$. Then*

$$B(\phi, \tilde{\sigma}_1\phi) = -3 \left(\sum_{i=1}^3 \left(\left(\sum_{j=0}^7 (D_i e_j | D_i e_j) \right) + 24(a_i | a_i) \right) \right).$$

(3) *$\tilde{\sigma}_1$ is a Cartan involution of $\mathfrak{f}_{4(-20)}$.*

Proof. (2) By Lemma 4.1(1), $\tilde{\sigma}_1\phi = d\varphi(D_1, D_2, D_3) + \sum_{i=1}^3 \epsilon_1(i)\tilde{A}_i^1(a_i)$. Since $\{E_i, F_i^1(e_j) \mid i = 1, 2, 3, j = 0, \dots, 7\}$ is a basis of \mathcal{J}^1 ,

$$B(\phi, \tilde{\sigma}_1\phi) = 3 \left(\sum_{i=1}^3 (\phi(\tilde{\sigma}_1\phi)E_i)_{E_i} + \sum_{i=1}^3 \sum_{j=0}^7 ((\phi(\tilde{\sigma}_1\phi)F_i^1(e_j))_{F_i^1} | e_j) \right).$$

follows from (1). Because of Lemmas 3.3(2) and 3.6(2), and direct calculation, we obtain $(\phi(\tilde{\sigma}_1\phi)E_i)_{E_i} = -2((a_{i+1}|a_{i+1}) + (a_{i+2}|a_{i+2}))$ and $((\phi(\tilde{\sigma}_1\phi)F_i^1(e_j))_{F_i^1} | e_j) = -(D_i e_j | D_i e_j) - 4(a_i | e_j)^2 - (a_{i+1}|a_{i+1}) - (a_{i+2}|a_{i+2})$ where indices $i, i+1, i+2$ are counted modulo 3. Thus (2) follows.

(3) Let $\phi \in \mathfrak{f}_{4(-20)}$. By Lemmas 3.6(1) and 3.3(3), ϕ can be expressed by $\phi = d\varphi(D_1, D_2, D_3) + \sum_{i=1}^3 \tilde{A}_i^1(a_i)$ where $d\varphi(D_1, D_2, D_3) \in \mathfrak{d}_4$ and $a_i \in \mathbf{O}$. By (2), $B(\phi, \tilde{\sigma}_1\phi) \leq 0$, and it follows that $B(\phi, \tilde{\sigma}_1\phi) = 0$ iff $\phi = 0$. Hence (3) follows. \square

Let us denote

$$\mathfrak{k} := \{\phi \in \mathfrak{f}_{4(-20)} \mid \tilde{\sigma}_1\phi = \phi\} = \text{Lie}(K), \quad \mathfrak{p} := \{\phi \in \mathfrak{f}_{4(-20)} \mid \tilde{\sigma}_1\phi = -\phi\}.$$

Then $\mathfrak{f}_{4(-20)} = \mathfrak{k} \oplus \mathfrak{p}$ (Cartan decomposition). By Lemma 4.1(1)(iii), $\tilde{A}_3^1(1) \in \mathfrak{p}$. The abelian subspace \mathfrak{a} of \mathfrak{p} , the analytic subgroup A of $F_{4(-20)}$ and the linear functional α on \mathfrak{a} are defined as

$$\mathfrak{a} := \{t\tilde{A}_3^1(1) \mid t \in \mathbb{R}\}, \quad A := \{\exp(t\tilde{A}_3^1(1)) \mid t \in \mathbb{R}\}, \quad \alpha(\tilde{A}_3^1(1)) := 1 \text{ (resp.)}.$$

Let $\mathfrak{a}_{\mathfrak{p}}$ be a maximal abelian subspace \mathfrak{a} of \mathfrak{p} such that $\mathfrak{a} \subset \mathfrak{a}_{\mathfrak{p}}$, and denote the dual space of \mathfrak{a} and $\mathfrak{a}_{\mathfrak{p}}$ as \mathfrak{a}^* and $\mathfrak{a}_{\mathfrak{p}}^*$, respectively. For $\lambda \in \mathfrak{a}^*$ or $\mathfrak{a}_{\mathfrak{p}}^*$,

$$\mathfrak{g}_{\lambda} := \{\phi \in \mathfrak{f}_{4(-20)} \mid [H, \phi] = \lambda(H)\phi \text{ for all } H \in \mathfrak{a}\},$$

$$\mathfrak{g}_{\lambda}(\mathfrak{a}_{\mathfrak{p}}) := \{\phi \in \mathfrak{f}_{4(-20)} \mid [H, \phi] = \lambda(H)\phi \text{ for all } H \in \mathfrak{a}_{\mathfrak{p}}\}$$

respectively. Moreover, let us denote

$$\Sigma := \{\lambda \in \mathfrak{a}^* \mid \lambda \neq 0, \mathfrak{g}_{\lambda} \neq \{0\}\},$$

$$\Sigma(\mathfrak{a}_{\mathfrak{p}}) := \{\lambda \in \mathfrak{a}_{\mathfrak{p}}^* \mid \lambda \neq 0, \mathfrak{g}_{\lambda}(\mathfrak{a}_{\mathfrak{p}}) \neq \{0\}\},$$

and the centralizer of \mathfrak{a} of the group K and its Lie algebra as

$$M := Z_K(\mathfrak{a}) = \{k \in K \mid k\tilde{A}_3^1(1)k^{-1} = \tilde{A}_3^1(1)\},$$

$$\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a}) = \{\phi \in \mathfrak{k} \mid [\phi, \tilde{A}_3^1(1)] = 0\}.$$

For all $p \in \text{Im}\mathbf{O}$, the elements $l_p, r_p, t_p \in \text{End}_{\mathbb{R}}(\mathbf{O})$ are defined as

$$l_px := px, \quad r_px := xp, \quad t_px := (l_p + r_p)x = px + xp \quad \text{for } x \in \mathbf{O}$$

respectively. Since $\bar{p} = -p$ and Lemma 1.1(4), $(l_px|y) = -(x|l_py)$, so that $D_1 \in \tilde{\mathfrak{d}}_4$. Similarly $l_p, t_p \in \tilde{\mathfrak{d}}_4$. By Lemma 1.1(9), $l_p(x)y + x r_p(y) = (px)y + x(y p) = p(xy) + (xy)p = \epsilon t_{-p}\epsilon(xy)$. Applying Lemma 3.3(3), the element $\delta(p) \in \mathfrak{d}_4$ is defined by

$$\delta(p) := d\varphi_0(l_p, r_p, t_{-p}).$$

For $p \in \text{Im}\mathbf{O}$ and $x \in \mathbf{O}$, let us denote

$$(4.1) \quad \mathcal{G}_1(x) := \tilde{A}_1^1(x) + \tilde{A}_2^1(-\bar{x}), \quad \mathcal{G}_2(p) := -\tilde{A}_3^1(p) - \delta(p),$$

$$\mathcal{G}_{-1}(x) := \tilde{A}_1^1(x) + \tilde{A}_2^1(\bar{x}), \quad \mathcal{G}_{-2}(p) := \tilde{A}_3^1(p) - \delta(p).$$

For $i = \pm 1$ and $j = \pm 2$, let us denote the subspaces \mathfrak{g}_i and \mathfrak{g}_j of $\mathfrak{f}_{4(-20)}$:

$$\mathfrak{g}_i := \{\mathcal{G}_i(p) \mid p \in \text{Im}\mathbf{O}\}, \quad \mathfrak{g}_j := \{\mathcal{G}_j(x) \mid x \in \mathbf{O}\}.$$

Lemma 4.3. *Let $p \in \text{Im}\mathbf{O}$ and $x \in \mathbf{O}$.*

(1) $\mathfrak{g}_i \subset \mathfrak{g}_{i\alpha}$ for $i \in \{\pm 1, \pm 2\}$. Especially, $\{\pm\alpha, \pm 2\alpha\} \subset \Sigma$.

(2) $[\mathfrak{g}_{i\alpha}, \mathfrak{g}_{j\alpha}] = \mathfrak{g}_{(i+j)\alpha}$.

Proof. (1) Since Lemma 4.1(2)(iii) and direct calculation,

$$[\tilde{A}_3^1(1), \tilde{A}_1^1(x) + \tilde{A}_2^1(\mp \bar{x})] = \pm(\tilde{A}_1^1(x) + \tilde{A}_2^1(\mp \bar{x})) \quad \text{for all } x \in \mathbf{O} \quad (\text{resp}).$$

Thus $\mathfrak{g}_{\pm 1} \subset \mathfrak{g}_{\pm \alpha}$ (resp). Fix $p \in \text{Im } \mathbf{O}$. Let $X \in \{E_i, F_i^1(x) | x \in \mathbf{O}\}$. Since Lemmas 4.1(2), 3.6(2), 3.3(1) and direct calculation,

$$[\tilde{A}_3^1(1), \tilde{A}_3^1(p)]X = 2\delta(p)X.$$

Thus $[\tilde{A}_3^1(1), \tilde{A}_3^1(p)] = 2\delta(p)$. Therefore

$$[\tilde{A}_3^1(1), -\tilde{A}_3^1(\pm p) - \delta(p)] = \pm 2(-\tilde{A}_3^1(\pm p) - \delta(p)) \quad (\text{resp})$$

and so $\mathfrak{g}_{\pm 2} \subset \mathfrak{g}_{\pm 2\alpha}$ (resp). Hence (1) follows.

(2) The assertion follows from the Jacobi identity. \square

Proposition 4.4. $M = B_3 = \varphi_0(\text{Spin}(7))$.

Proof. By Lemma 3.2(2), Lemma 3.1(2) and Proposition 3.16(2), note that $\varphi_0(\text{Spin}(7)) = B_3 = (F_{4(-20)})_{E_1, F_3^1(1)}$ and $K = (F_{4(-20)})_{E_1}$. Let $g \in \text{Spin}(7)$. g can be expressed by $g = (g_1, g_2, g_3)$ such that $g = (g_1, \epsilon g_1 \epsilon, g_3) \in \text{Spin}(8)$ and $g_3 1 = 1$. Put $\phi = \varphi_0(g) \tilde{A}_3^1(1) \varphi_0(g)^{-1}$. Since Lemmas 3.4, 3.6(2) and direct calculation,

$$\begin{aligned} \phi(-E_1 + E_2) &= 2F_3^1(1) = \tilde{A}_3^1(1)(-E_1 + E_2), \\ \phi P^- &= 2P^- = \tilde{A}_3^1(1)P^-, \quad \phi E = 0 = \tilde{A}_3^1(1)E, \quad \phi E_3 = 0 = \tilde{A}_3^1(1)E_3, \\ \phi F_3^1(p) &= 2(g_3^{-1} p | 1)(-E_1 + E_2) = 2(p | 1)(-E_1 + E_2) = \tilde{A}_3^1(1)F_3^1(p), \\ \phi Q^+(x) &= Q^+(\bar{x}) = \tilde{A}_3^1(1)Q^+(x), \quad \phi Q^-(x) = -Q^-(\bar{x}) = \tilde{A}_3^1(1)Q^-(x) \end{aligned}$$

where $x \in \mathbf{O}$ and $p \in \text{Im } \mathbf{O}$. Therefore $\phi = \tilde{A}_3^1(1)$, and so $\varphi_0(g) \in M$. Thus $B_3 = \varphi_0(\text{Spin}(7)) \subset M$.

Conversely, take $k \in M$. Then $k \tilde{A}_3^1(1) k^{-1} = \tilde{A}_3^1(1)$. We consider the equation $k \tilde{A}_3^1(1) k^{-1} E_1 = \tilde{A}_3^1(1) E_1$. Since $k^{-1} \in (F_{4(-20)})_{E_1}$ and Lemma 3.6(2), the right hand side is $k \tilde{A}_3^1(1) k^{-1} E_1 = -k F_3^1(1)$, and the left hand side is $\tilde{A}_3^1(1) E_1 = -F_3^1(1)$. Therefore $k F_3^1(1) = F_3^1(1)$ and so $k \in (F_{4(-20)})_{E_1, F_3^1(1)} = B_3$. Hence $M \subset B_3$ and so $M = B_3 = \varphi_0(\text{Spin}(7))$. \square

Lemma 4.5. Let $i \in \{\pm 1, \pm 2\}$.

(1) $\mathfrak{g}_i = \mathfrak{g}_{i\alpha}$. (2) $\mathfrak{a}_p = \mathfrak{a}$. (3) $\Sigma(\mathfrak{a}_p) = \Sigma = \{\pm \alpha, \pm 2\alpha\}$.

(4) $\mathfrak{f}_{4(-20)} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$.

Proof. By virtue of definitions of \mathfrak{m} and \mathfrak{a} , $\mathfrak{m} \subset \mathfrak{g}_0 \cap \mathfrak{k}$ and $\mathfrak{a} \subset \mathfrak{g}_0 \cap \mathfrak{p}$. By Lemma 4.3(1), $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{a} + \mathfrak{m} + \mathfrak{g}_1 + \mathfrak{g}_2$ is a direct sum $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \subset \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \subset \mathfrak{f}_{4(-20)}$. Next $\dim \mathfrak{a} = 1$, $\dim \mathfrak{g}_2 = \dim \mathfrak{g}_{-2} = \dim \mathbf{O} = 8$ and $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_{-1} = \dim (\text{Im } \mathbf{O}) = 7$. By Proposition 4.4 and Lemma 3.2(2), $\dim \mathfrak{m} = \dim (\mathfrak{so}(7)) = 21$ where

$\mathfrak{so}(7) := \text{Lie}(\text{SO}(7))$. Moreover, $\dim \mathfrak{f}_{4(-20)} = 52$ by Lemma 3.6(1). Thus $\dim (\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2) = 52 = \dim \mathfrak{f}_{4(-20)}$ and so

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} = \mathfrak{f}_{4(-20)}.$$

By Lemma 4.3(1), $\mathfrak{g}_i = \mathfrak{g}_{i\alpha}$ for $i \in \{\pm 1, \pm 2\}$. Because of $\mathfrak{a} \oplus \mathfrak{m} \subset \mathfrak{g}_0$, the decomposition $\mathfrak{f}_{4(-20)} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus (\mathfrak{a} \oplus \mathfrak{m}) \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ implies the eigendecomposition of $\text{ad}(\tilde{A}_3^1(1))$ and the root space decomposition of $(\mathfrak{f}_{4(-20)}, \mathfrak{a})$ (cf. [18, Ch V]). Therefore $\Sigma = \{\pm\alpha, \pm 2\alpha\}$ and $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$. Because of $\mathfrak{a} \subset \mathfrak{a}_{\mathfrak{p}} \subset \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$, \mathfrak{a} is a maximal abelian subalgebra of \mathfrak{p} . Therefore $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{a}$, and so $\Sigma(\mathfrak{a}_{\mathfrak{p}}) = \Sigma = \{\pm\alpha, \pm 2\alpha\}$. Hence the assertion follows. \square

The nilpotent subalgebras \mathfrak{n}^\pm are defined as

$$\mathfrak{n}^+ := \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_\alpha = \{\mathcal{G}_2(p) + \mathcal{G}_1(x) \mid p \in \text{Im } \mathbf{O}, x \in \mathbf{O}\},$$

$$\mathfrak{n}^- := \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} = \{\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x) \mid p \in \text{Im } \mathbf{O}, x \in \mathbf{O}\}$$

respectively. In fact, by Lemma 4.5(1), $\mathfrak{g}_{\pm 2\alpha} \oplus \mathfrak{g}_{\pm\alpha} = \{\mathcal{G}_{\pm 2}(p) + \mathcal{G}_{\pm 1}(x) \mid p \in \text{Im } \mathbf{O}, x \in \mathbf{O}\}$ (resp), and by Lemmas 4.3(2) and 4.5(3),

$$[\mathfrak{n}^+, [\mathfrak{n}^+, \mathfrak{n}^+]] = 0, \quad [\mathfrak{n}^-, [\mathfrak{n}^-, \mathfrak{n}^-]] = 0$$

and the nilpotent subgroups N^\pm of $F_{4(-20)}$ are defined as

$$N^+ := \exp \mathfrak{n}^+ = \{\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) \mid p \in \text{Im } \mathbf{O}, x \in \mathbf{O}\},$$

$$N^- := \exp \mathfrak{n}^- = \{\exp(\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x)) \mid p \in \text{Im } \mathbf{O}, x \in \mathbf{O}\} \text{ (resp)}.$$

Lemma 4.6. *Let $x \in \mathbf{O}$ and $p \in \text{Im } \mathbf{O}$.*

$$(1) \exp \mathcal{G}_2(p) \exp \mathcal{G}_1(x) = \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) = \exp \mathcal{G}_1(x) \exp \mathcal{G}_2(p).$$

$$(2) \tilde{\sigma}_1(\mathcal{G}_{\pm 2}(p) + \mathcal{G}_{\pm 1}(x)) = \mathcal{G}_{\mp 2}(p) + \mathcal{G}_{\mp 1}(x) \text{ (resp)}. \text{ Especially, } \tilde{\sigma}_1 \mathfrak{n}^+ = \mathfrak{n}^- \text{ and } \tilde{\sigma}_1 \mathfrak{n}^- = \mathfrak{n}^+.$$

$$(3) \tilde{\sigma}_1 \exp(\mathcal{G}_{\pm 2}(p) + \mathcal{G}_{\pm 1}(p)) = \exp(\mathcal{G}_{\mp 2}(p) + \mathcal{G}_{\mp 1}(p)) \text{ (resp)}. \text{ Especially, } \tilde{\sigma}_1(N^+) = \sigma_1 N^+ \sigma_1 = N^- \text{ and } \tilde{\sigma}_1(N^-) = \sigma_1 N^- \sigma_1 = N^+.$$

Proof. (1) By Lemmas 4.3(2) and 4.5(3), $[\mathfrak{g}_\alpha, \mathfrak{g}_{2\alpha}] = [\mathfrak{g}_{2\alpha}, \mathfrak{g}_\alpha] = 0$. Hence (1) follows.

(2) By Lemma 4.1(1), $\tilde{\sigma}_1(\mathcal{G}_{\pm 2}(p) + \mathcal{G}_{\pm 1}(x)) = \mathcal{G}_{\mp 2}(p) + \mathcal{G}_{\mp 1}(x)$ (resp). Hence (2) follows.

(3) By (2), $\tilde{\sigma}_1 \exp(\mathcal{G}_{\pm 2}(p) + \mathcal{G}_{\pm 1}(p)) = \exp(\tilde{\sigma}_1(\mathcal{G}_{\pm 2}(p) + \mathcal{G}_{\pm 1}(p))) = \exp(\mathcal{G}_{\mp 2}(p) + \mathcal{G}_{\mp 1}(p))$ (resp). Hence (3) follows. \square

For $x, y, z \in \mathbf{O}$, let us denote $x \times y \times z := \frac{1}{2}(x(\bar{y}z) - z(\bar{y}x))$ and

$$T(x, y, z) := -\overline{x \times y \times z} = \frac{1}{2}((x\bar{y})z - (z\bar{y})x).$$

Lemma 4.7. *$T(x, y, z)$ is alternating on \mathbf{O} .*

Proof. By [13, Lemma 6.56(a)], $x \times y \times z$ is alternating on \mathbf{O} . Hence $T(x, y, z)$ is alternating on \mathbf{O} . \square

Lemma 4.8. *Let $p, q \in \text{Im } \mathbf{O}$ and $x, y \in \mathbf{O}$. Then*

$$[\mathcal{G}_2(p) + \mathcal{G}_1(x), \mathcal{G}_2(q) + \mathcal{G}_1(y)] = \mathcal{G}_2(2\text{Im}(x\bar{y})).$$

Proof. It is enough to show $[\mathcal{G}_2(p), \mathcal{G}_2(q)] = [\mathcal{G}_2(p), \mathcal{G}_1(x)] = 0$ and $[\mathcal{G}_1(x), \mathcal{G}_1(y)] = \mathcal{G}_2(2\text{Im}(x\bar{y}))$. At first, by Lemmas 4.5(3) and 4.3(2),

$$[\mathcal{G}_2(p), \mathcal{G}_2(q)] = [\mathcal{G}_2(p), \mathcal{G}_1(x)] = 0.$$

At second, by Lemma 4.1(2) and direct calculation,

$$[\mathcal{G}_1(x), \mathcal{G}_1(y)] = ([\tilde{A}_1^1(x), \tilde{A}_1^1(y)] + [\tilde{A}_2^1(\bar{x}), \tilde{A}_2^1(\bar{y})]) - \tilde{A}_3^1(2\text{Im}(x\bar{y})).$$

Put $f = [\mathcal{G}_1(x), \mathcal{G}_1(y)] - \mathcal{G}_2(2\text{Im}(x\bar{y})) = [\tilde{A}_1^1(\bar{x}), \tilde{A}_1^1(\bar{y})] + [\tilde{A}_2^1(x), \tilde{A}_2^1(y)] + \delta(2\text{Im}(x\bar{y}))$. To prove $[\mathcal{G}_1(x), \mathcal{G}_1(y)] = \mathcal{G}_2(2\text{Im}(x\bar{y}))$, it is enough to show $f = 0$. By Lemma 4.1(2)(ii), $f \in \mathfrak{d}_4$, and so by Lemma 3.3(3), $f = d\varphi_0(D_1, D_2, D_3)$ for some $(D_1, D_2, D_3) \in (\tilde{\mathfrak{d}}_4)^3$ satisfying the infinitesimal triality: $(D_1x)y + x(D_2y) = \epsilon D_3\epsilon(xy)$ for all $x, y \in \mathbf{O}$. Fix $z \in \mathbf{O}$. Then $F_1^1(D_1z) = d\varphi_0(D_1, D_2, D_3)F_2^1(z) = fF_2^1(z) = ([\tilde{A}_1^1(x), \tilde{A}_1^1(y)] + [\tilde{A}_2^1(\bar{x}), \tilde{A}_2^1(\bar{y})] + \delta(2\text{Im}(x\bar{y})))F_1^1(z)$. Since Lemmas 3.6(2), 1.1(6), and 4.7 and direct calculation,

$$\begin{aligned} D_1z &= -4(y|z)x + 4(x|z)y + (z\bar{y})x - (z\bar{x})y + (x\bar{y})z - (y\bar{x})z \\ &= -2(y\bar{z})z - 2(z\bar{y})x + 2(x\bar{z})y + 2(z\bar{x})y \\ &\quad + (z\bar{y})x - (z\bar{x})y + (x\bar{y})z - (y\bar{x})z \\ &= (z\bar{x})y - (y\bar{x})z + (x\bar{y})z - (z\bar{y})x + 2(x\bar{z})y - 2(y\bar{z})x \\ &= 2T(z, x, y) + 2T(x, y, z) + 4T(x, z, y) \\ &= 2T(x, y, z) + 2T(x, y, z) - 4T(x, y, z) = 0 \end{aligned}$$

Therefore $F_1(D_1z) = 0$ and so $D_1 = 0$. Using Lemma 3.3(1), $D_2 = D_3 = 0$ and so $f = 0$. Hence the assertion follows. \square

Lemma 4.9. *There exists a neighborhood U of 0 in $\text{Im } \mathbf{O} \times \mathbf{O}$ such that*

$$\begin{aligned} &\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) \exp(\mathcal{G}_2(q) + \mathcal{G}_1(y)) \\ &= \exp(\mathcal{G}_2(p + q + \text{Im}(x\bar{y})) + \mathcal{G}_1(x + y)) \\ &= \exp(\mathcal{G}_2(p + q + \text{Im}(x\bar{y}))) \exp(\mathcal{G}_1(x + y)) \quad \text{for all } (p, x), (q, y) \in U. \end{aligned}$$

Proof. (1) Using Campbell-Hausdorff-Dynkin formulas (cf. [8, Theorem 3.4.4]), there exists a neighborhood U_1 of 0 in $\text{End}_{\mathbb{R}}(\mathcal{J}^1)$ such that for all $X, Y \in U_1$,

$$\begin{aligned} \exp X \exp Y &= \exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] \\ &\quad + (\text{terms of degree } \geq 4)). \end{aligned}$$

Because of $[\mathfrak{n}^+, [\mathfrak{n}^+, \mathfrak{n}^+]] = 0$,

$$\exp X \exp Y = \exp(X + Y + \frac{1}{2}[X, Y]) \quad \text{for all } X, Y \in \mathfrak{n}^+ \cap U_1.$$

Therefore, there exists a neighborhood U of 0 in $\text{Im}\mathbf{O} \times \mathbf{O}$ such that for all $(p, x), (q, y) \in U$,

$$\begin{aligned} & \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) \exp(\mathcal{G}_2(q) + \mathcal{G}_1(y)) \\ &= \exp(\mathcal{G}_2(p+q) + \mathcal{G}_1(x+y) + \frac{1}{2}[\mathcal{G}_2(p) + \mathcal{G}_1(x), \mathcal{G}_2(q) + \mathcal{G}_1(y)]) \end{aligned}$$

Applying Lemma 4.8,

$$\begin{aligned} & \exp(\mathcal{G}_2(p+q) + \mathcal{G}_1(x+y) + \frac{1}{2}[\mathcal{G}_2(p) + \mathcal{G}_1(x), \mathcal{G}_2(q) + \mathcal{G}_1(y)]) \\ &= \exp(\mathcal{G}_2(p+q + \text{Im}(x\bar{y})) + \mathcal{G}_1(x+y)). \end{aligned}$$

By Lemma 4.6(1), $\exp(\mathcal{G}_2(p+q + \text{Im}(x\bar{y})) + \mathcal{G}_1(x+y)) = \exp(\mathcal{G}_2(p+q + \text{Im}(x\bar{y})) \exp(\mathcal{G}_1(x+y)))$. \square

Lemma 4.10. *Let $p, q \in \text{Im}\mathbf{O}$ and $x, y \in \mathbf{O}$.*

(1) *The following equations hold.*

$$\left\{ \begin{array}{ll} \text{(i)} & \mathcal{G}_2(p)(-E_1 + E_2) = -2F_3^1(p), \quad \text{(ii)} \quad \mathcal{G}_2(p)P^- = 0, \\ \text{(iii)} & \mathcal{G}_2(p)E = 0, \quad \text{(iv)} \quad \mathcal{G}_2(p)E_3 = 0, \\ \text{(v)} & \mathcal{G}_2(p)F_3^1(q) = -2(p|q)P^-, \\ \text{(vi)} & \mathcal{G}_2(p)Q^+(y) = 0, \\ \text{(vii)} & \mathcal{G}_2(p)Q^-(y) = -2Q^+(py). \end{array} \right.$$

(2) *The following equations hold.*

$$\left\{ \begin{array}{ll} \text{(i)} & \mathcal{G}_1(x)(-E_1 + E_2) = -Q^-(x), \quad \text{(ii)} \quad \mathcal{G}_1(x)P^- = 0, \\ \text{(iii)} & \mathcal{G}_1(x)E = 0, \quad \text{(iv)} \quad \mathcal{G}_1(x)E_3 = Q^+(x), \\ \text{(v)} & \mathcal{G}_1(x)F_3^1(q) = -Q^+(qx), \\ \text{(vi)} & \mathcal{G}_1(x)Q^+(y) = 2(x|y)P^-, \\ \text{(vii)} & \mathcal{G}_1(x)Q^-(y) = 2(x|y)(E - 3E_3) + F_3^1(2\text{Im}(x\bar{y})). \end{array} \right.$$

Proof. It follows from Lemma 3.6(2), the definition of $\delta(p)$ and direct calculation. \square

Lemma 4.11. *For all $X \in \mathfrak{n}^+$, $\mathcal{G}_2(p)^3 = 0$, $\mathcal{G}_1(x)^5 = 0$ and $X^8 = 0$. Especially, X is a nilpotent element of $\text{End}_{\mathbb{R}}(\mathcal{J}^1)$ and*

$$\exp \mathcal{G}_2(p) = \sum_{n=0}^2 \frac{1}{n!} \mathcal{G}_2(p)^n, \quad \exp \mathcal{G}_1(x) = \sum_{n=0}^4 \frac{1}{n!} \mathcal{G}_1(x)^n.$$

Proof. Since Lemma 4.10 and direct calculation, for all $Y \in \{-E_1 + E_2, P^-, E, E_3, F_3^1(p), Q^+(x), Q^-(y) \mid p \in \text{Im}\mathbf{O}, x, y \in \mathbf{O}\}$, $\mathcal{G}_2(p)^3 Y = 0$ and $\mathcal{G}_1(x)^5 Y = 0$. Thus $\mathcal{G}_2(p)^3 = 0$ and $\mathcal{G}_1(x)^5 = 0$ by Lemma 1.2(2). Put $X = \mathcal{G}_2(p) + \mathcal{G}_1(x)$. By Lemma 4.8, $[\mathcal{G}_2(p), \mathcal{G}_1(x)] = 0$. Therefore, using the binomial theorem, $X^8 = \sum_{k=0}^8 \binom{8}{k} \mathcal{G}_2(p)^{8-k} \mathcal{G}_1(x)^k$. Because of $8-k \geq 3$ or $k \geq 5$, $X^8 = 0$. \square

Proposition 4.12. *Let $p, q \in \text{Im}\mathbf{O}$ and $x, y \in \mathbf{O}$.*

(1) *The following equations hold.*

$$\left\{ \begin{array}{l} \text{(i)} \quad \exp \mathcal{G}_2(p)(-E_1 + E_2) = (-E_1 + E_2) - 2F_3^1(p) + 2(p|p)P^-, \\ \text{(ii)} \quad \exp \mathcal{G}_2(p)P^- = P^-, \\ \text{(iii)} \quad \exp \mathcal{G}_2(p)E = E, \quad \text{(iv)} \quad \exp \mathcal{G}_2(p)E_3 = E_3, \\ \text{(v)} \quad \exp \mathcal{G}_2(p)F_3^1(q) = F_3^1(q) - 2(p|q)P^-, \\ \text{(vi)} \quad \exp \mathcal{G}_2(p)Q^+(y) = Q^+(y), \\ \text{(vii)} \quad \exp \mathcal{G}_2(p)Q^-(y) = Q^-(y) - 2Q^+(py). \end{array} \right.$$

(2) *The following equations hold.*

$$\left\{ \begin{array}{l} \text{(i)} \quad \exp \mathcal{G}_1(x)(-E_1 + E_2) = (-E_1 + E_2) - Q^-(x) \\ \quad \quad \quad - (x|x)(E - 3E_3) + (x|x)Q^+(x) + \frac{1}{2}(x|x)^2P^-, \\ \text{(ii)} \quad \exp \mathcal{G}_1(x)P^- = P^-, \quad \text{(iii)} \quad \exp \mathcal{G}_1(x)E = E, \\ \text{(iv)} \quad \exp \mathcal{G}_1(x)E_3 = E_3 + Q^+(x) + (x|x)P^-, \\ \text{(v)} \quad \exp \mathcal{G}_1(x)F_3^1(q) = F_3^1(q) - Q^+(qx), \\ \text{(vi)} \quad \exp \mathcal{G}_1(x)Q^+(y) = Q^+(y) + 2(x|y)P^-, \\ \text{(vii)} \quad \exp \mathcal{G}_1(x)Q^-(y) = Q^-(y) + 2(x|y)(E - 3E_3) \\ \quad \quad \quad + F_3^1(2\text{Im}(x\bar{y})) - Q^+(3(x|y)x + \text{Im}(x\bar{y})x) - 2(x|y)(x|x)P^-. \end{array} \right.$$

Proof. It follows from Lemma 4.10 and direct calculation. \square

5. THE STABILIZER GROUPS OF SEMIDIRECT PRODUCT GROUP TYPE.

Consider $\text{Spin}(7) \times \text{Im}\mathbf{O} \times \mathbf{O}$ in which multiplication is defined by

$$(5.1) \quad (g, p, x)(h, q, y) := (gh, p + g_3q + \text{Im}(x\overline{(g_1y)}), x + g_1y)$$

where $p, q \in \text{Im}\mathbf{O}$, $x, y \in \mathbf{O}$ and $g = (g_1, g_2, g_3), h \in \text{Spin}(7)$. Then the multiplication is closed because of $\text{Im}(x\overline{(g_1y)})$, $g_3q \in \text{Im}\mathbf{O}$ by Lemma 2.3(1)(iv). Let us denote $G := \text{Spin}(7) \times \text{Im}\mathbf{O} \times \mathbf{O}$, and the subsets H, N of G as

$$H := \{(g, 0, 0) \mid g \in \text{Spin}(7)\}, \quad N := \{(1, p, x) \mid p \in \text{Im}\mathbf{O}, x \in \mathbf{O}\}.$$

Lemma 5.1.

- (1) G is a group with respect to the multiplication.
- (2) H and N are subgroups of G ; $H \cong \text{Spin}(7)$ and $N \cong \text{Im}\mathbf{O} \times \mathbf{O}$.
- (3) G is the semi-direct product $H \ltimes N$.

Proof. (1) Let $g = (g_1, g_2, g_3), h = (h_1, h_2, h_3), f \in \text{Spin}(7)$, $p, q, r \in \text{Im}\mathbf{O}$ and $x, y, z \in \mathbf{O}$. Then $g_3(\text{Im}(y\overline{(h_1z)})) = \text{Im}((g_1y)\overline{(g_1h_1z)})$ by

Lemma 2.3(3)(ii). Therefore

$$\begin{aligned}
& ((g, p, x)(h, q, y))(f, r, z) \\
&= (ghf, p + g_3q + g_3h_3r + \text{Im}(x\overline{(g_1y)})) \\
&\quad + \text{Im}(x\overline{(g_1h_1z)}) + \text{Im}((g_1y)\overline{(g_1h_1z)}), x + g_1y + g_1h_1z) \\
&= (ghf, p + g_3q + g_3h_3r + \text{Im}(x\overline{(g_1y)})) \\
&\quad + \text{Im}(x\overline{(g_1h_1z)}) + g_3(\text{Im}(y\overline{(h_1z)})), x + g_1y + g_1h_1z) \\
&= (g, p, x)((h, q, y)(f, r, z))
\end{aligned}$$

and so the associativity hold. The identity element is $(1, 0, 0)$ and the inverse element of (g, p, x) is $(g^{-1}, -g_3^{-1}p, -g_1^{-1}x)$. Hence (1) follows.

(2) $(g, 0, 0)^{-1}(h, 0, 0) = (g^{-1}h, 0, 0) \in H$ and $(1, p, x)^{-1}(1, q, y) = (1, *, *) \in N$, so that H and N are subgroups of G . Obviously isomorphisms follow. Hence (2) follows.

(3) $G = HN$ follows from $(g, p, x) = (g, 0, 0)(1, g_3^{-1}p, g_1^{-1}x)$ where $g = (g_1, g_2, g_3) \in \text{Spin}(7)$. Obviously $H \cap N = \{(1, 0, 0)\}$ follows. For any $g \in G$, $(g, p, x)(1, q, y)(g^{-1}, -g_3^{-1}p, -g_1^{-1}x) = (1, *, *) \in N$, and so $gNg^{-1} \subset N$. Therefore N is a normal subgroup of G . Hence $G = H \ltimes N$ follows. \square

From now on, let us denote $H = \text{Spin}(7)$, $N = \text{Im}\mathbf{O} \times \mathbf{O}$, and $G = \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})$. Let us denote

$$\begin{aligned}
G' &:= \{(g, p, 0) \mid g \in \text{Spin}(7), p \in \text{Im}\mathbf{O}\}, \\
G'' &:= \{(g, p, x) \mid g \in G_2, p, x \in \text{Im}\mathbf{O}\}, \\
N' &:= \{(1, p, 0) \mid p \in \text{Im}\mathbf{O}\} \subset G', \\
H'' &:= \{(g, 0, 0) \mid g \in G_2\} \subset G'', \\
N'' &:= \{(1, p, q) \mid p, q \in \text{Im}\mathbf{O}\} \subset G''.
\end{aligned}$$

Then we have:

Lemma 5.2.

- (1) G' and G'' are subgroups of $\text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})$.
- (2) H and N' are subgroups of G' ; $N' \cong \text{Im}\mathbf{O}$.
- (3) H'' and N'' are subgroups of G'' ; $H'' \cong G_2$, $N'' \cong \text{Im}\mathbf{O} \times \text{Im}\mathbf{O}$.
- (4) G' is the semi-direct product $H \ltimes N'$.
- (5) G'' is the semi-direct product $H'' \ltimes N''$.

From now on, let us denote

$$\begin{aligned}
\text{Im}\mathbf{O} &:= N', \quad G_2 := H'', \quad \text{Im}\mathbf{O} \times \text{Im}\mathbf{O} := N'', \\
\text{Spin}(7) \ltimes \text{Im}\mathbf{O} &:= G', \quad G_2 \ltimes (\text{Im}\mathbf{O} \times \text{Im}\mathbf{O}) := G''.
\end{aligned}$$

The map $\varphi : \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O}) \rightarrow (F_{4(-20)})_{P^-}$ is defined by

$$(5.2) \quad \begin{aligned} \varphi(g, p, x) &:= \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))\varphi_0(g) \\ &= \exp \mathcal{G}_2(p) \exp \mathcal{G}_1(x)\varphi_0(g) = \exp \mathcal{G}_1(x) \exp \mathcal{G}_2(p)\varphi_0(g), \\ &\text{for } (g, p, x) \in \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O}) \end{aligned}$$

(cf. Lemma 4.6(1)). Then by Lemma 3.4 and Proposition 4.12, we get $\varphi(g, p, x)P^- = P^-$, and so the mapping is well-defined.

Proposition 5.3. *Let $g = (g_1, g_2, g_3) \in \text{Spin}(7)$, $p, q \in \text{Im}\mathbf{O}$ and $x, y \in \mathbf{O}$.*

- (1) $\varphi_0(g) \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))\varphi_0(g)^{-1} = \exp(\mathcal{G}_2(g_3p) + \mathcal{G}_1(g_1x))$.
- (2) $\exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) \exp(\mathcal{G}_2(q) + \mathcal{G}_1(y))$
 $= \exp(\mathcal{G}_2(p + q + \text{Im}(x\bar{y})) + \mathcal{G}_1(x + y))$.
- (3) φ is a group homomorphism.
- (4) $\varphi(\text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})) = N^+M \subset (F_{4(-20)})_{P^-}$.

Proof. (1) Put $A = \varphi_0(g)$. Because of $A \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x))A^{-1} = \exp(A(\mathcal{G}_2(p) + \mathcal{G}_1(x))A^{-1})$ and Lemma 1.2(2), it is enough to show that for all $X \in \{-E_1 + E_2, P^-, E, E_3, F_3^1(q), Q^+(y), Q^-(z) \mid q \in \text{Im}\mathbf{O}, y, z \in \mathbf{O}\}$,

$$(i) \quad A\mathcal{G}_2(p)A^{-1}X = \mathcal{G}_2(g_3p)X, \quad (ii) \quad A\mathcal{G}_1(x)A^{-1}X = \mathcal{G}_1(g_1x)X.$$

(Step 1) We will show (i) using Lemma 3.4 and Lemma 4.10(1).

Case $X = P^-, E, E_3, Q^+(y)$. Then $A\mathcal{G}_2(p)A^{-1}X = 0 = \mathcal{G}_2(g_3p)X$.

Case $X = -E_1 + E_2$. Then $A\mathcal{G}_2(p)A^{-1}(-E_1 + E_2) = -2F_3^1(g_3p) = \mathcal{G}_2(g_3p)(-E_1 + E_2)$.

Case $X = F_3^1(q)$. Since $(g_1, g_2, g_3) \in \text{Spin}(7)$, $A\mathcal{G}_2(p)A^{-1}F_3^1(q) = -2(p|g_3^{-1}q)P^- = -2(g_3p|q)P^- = \mathcal{G}_2(g_3p)F_3^1(q)$.

Case $X = Q^-(z)$. Then by Lemma 2.3(3)(iii), $A\mathcal{G}_2(p)A^{-1}Q^-(z) = -2Q^+(g_1(p|g_1^{-1}z))) = -2Q^+((g_3p|z)) = \mathcal{G}_2(g_3p)Q^-(z)$.

Therefore (i) follows.

(Step 2) We will show (ii) using Lemma 3.4 and Lemma 4.10(2).

Case $X = E, P^-$. Then $A\mathcal{G}_1(x)A^{-1}X = 0 = \mathcal{G}_1(g_1x)X$.

Case $X = -E_1 + E_2$. Then $A\mathcal{G}_1(x)A^{-1}(-E_1 + E_2) = -Q^-(g_1x) = \mathcal{G}_1(g_1x)(-E_1 + E_2)$.

Case $X = E_3$. Then $A\mathcal{G}_1(x)A^{-1}E_3 = Q^+(g_1x) = \mathcal{G}_1(g_1x)E_3$.

Case $X = Q^+(y)$. Since $(g_1, g_2, g_3) \in \text{Spin}(7)$, $A\mathcal{G}_1(x)A^{-1}Q^+(y) = -2(x|g_1^{-1}y)P^- = -2(g_1x|y)P^- = \mathcal{G}_2(g_1p)Q^+(y)$.

Case $X = F_3^1(q)$. Then by Lemma 2.3(3)(iii), $A\mathcal{G}_1(x)A^{-1}F_3^1(q) = F_3^1(g_1((g_3^{-1}q)x)) = F_3^1(q(g_1x)) = \mathcal{G}_1(g_1x)F_3^1(q)$.

Case $X = Q^-(z)$. Since $(g_1, g_2, g_3) \in \text{Spin}(7)$ and Lemma 2.3(3)(ii), $A \exp \mathcal{G}_1(x)A^{-1}Q^-(z) = 2(x|g_1^{-1}z)(E - 3E_3) + F_3^1(2g_3(\text{Im}x(g_1^{-1}z))) = 2(g_1x|z)(E - 3E_3) + F_3^1(2(g_1x|\bar{z})) = \mathcal{G}_1(g_1x)Q^-(z)$.

Therefore (ii) follows. Hence (2) follows.

(2) Put $f(p, x, q, y) \in \text{End}_{\mathbb{R}}(\mathcal{J}^1)$ as

$$f(p, x, q, y) = \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) \exp(\mathcal{G}_2(q) + \mathcal{G}_1(y)) \\ - \exp(\mathcal{G}_2(p + q + \text{Im}(x\bar{y})) + \mathcal{G}_1(x + y)).$$

Let p_i, q_i, x_i, y_i be variables defined as $p = \sum_{i=1}^7 p_i e_i$, $q = \sum_{i=1}^7 q_i e_i$, $x = \sum_{i=0}^7 x_i e_i$ and $y = \sum_{i=0}^7 y_i e_i$. Fix $X, Y \in \mathcal{J}^1$. Put the function $F_{X,Y}(p, x, q, y) = (f(p, x, q, y)X|Y)$. By Lemma 4.11,

$$f(p, x, q, y) = \left(\sum_{i=0}^2 \frac{1}{i!} \mathcal{G}_2(p)^i\right) \left(\sum_{i=0}^4 \frac{1}{i!} \mathcal{G}_1(x)^i\right) \left(\sum_{i=0}^2 \frac{1}{i!} \mathcal{G}_2(q)^i\right) \left(\sum_{i=0}^4 \frac{1}{i!} \mathcal{G}_1(y)^i\right) \\ - \left(\sum_{i=0}^2 \frac{1}{i!} \mathcal{G}_2(p + q + \text{Im}(x\bar{y}))^i\right) \left(\sum_{i=0}^4 \frac{1}{i!} \mathcal{G}_1(x + y)^i\right).$$

Thus $F_{X,Y}(p, x, q, y)$ is a polynomial function. By Lemma 4.9, there exists a neighborhood U of 0 in $(\text{Im}\mathbf{O} \times \mathbf{O})^2$ such that $f(p, x, q, y) = 0$ for all $(p, x, q, y) \in U$. Thus $F_{X,Y}(p, x, q, y) = 0$ for all $(p, x, q, y) \in U$. Since $F_{X,Y}(p, x, q, y)$ is a polynomial function, $F_{X,Y}(p, x, q, y) = 0$ for all $(p, x, q, y) \in (\text{Im}\mathbf{O} \times \mathbf{O})^2$. Moving $X, Y \in \mathcal{J}^1$, $f(p, x, q, y) = 0$. Hence (2) follows.

(3) Denote $F(p_0, x_0) = \exp(\mathcal{G}_2(p_0) + \mathcal{G}_1(x_0))$ for $(p_0, x_0) \in \text{Im}\mathbf{O} \times \mathbf{O}$. By (1),(2) and Lemma 3.2(2),

$$\varphi(g, p, x) \varphi(h, q, y) = F(p, x) (\varphi_0(g) F(q, y) \varphi_0(g)^{-1}) \varphi_0(gh) \\ = F(p, x) F(g_3 q, g_1 y) \varphi_0(gh) \\ = F(p + g_3 q + \text{Im}(x \overline{g_1 y}), x + g_1 y) \varphi_0(gh) \\ = \varphi(gh, p + g_3 q + \text{Im}(x \overline{g_1 y}), x + g_1 y).$$

Hence (3) follows.

(4) It follows from the definition of φ and Proposition 4.4. \square

Let V be a \mathbb{R} -linear space and N a nilpotent subgroup of $\text{GL}_{\mathbb{R}}(V)$. For $v \in V$, the subset $\text{Orb}_N(v)$ of V is called a *parabolic type plane*. Let us denote nilpotent subgroups N_1 and N_2 of $N^+ = \varphi(\text{Im}\mathbf{O} \times \mathbf{O})$ as

$$N_1 := \varphi(\text{Im}\mathbf{O}), \quad N_2 := \varphi(\text{Im}\mathbf{O} \times \text{Im}\mathbf{O})$$

and the subsets \mathcal{P}_{E_3, P^-} , \mathcal{P}_{P^-} and $\mathcal{P}_{Q^+(1)}$ of \mathcal{J}^1 as

$$\mathcal{P}_{E_3, P^-} := \{X \in (\mathcal{J}^1)_{2E_3, -1} \mid P^- \times X = -E_3, Q_{E_3}(X) = 1\}, \\ \mathcal{P}_{P^-} := \{X \in \mathcal{J}^1 \mid P^- \times X = -\frac{1}{2}P^-, X^{\times 2} = 0, \text{tr}(X) = 1\}, \\ \mathcal{P}_{Q^+(1)} := \{X \in \mathcal{P}_{P^-} \mid Q^+(1) \times X = 0\}.$$

Lemma 5.4.

$$\begin{aligned}
(1) \quad \mathcal{P}_{E_3, P^-} &= \{(-E_1 + E_2) + 2F_3^1(p) + 2(p|p)P^- \mid p \in \text{Im}\mathbf{O}\} \\
&= \{\exp \mathcal{G}_2(p)(-E_1 + E_2) \mid p \in \text{Im}\mathbf{O}\} \\
&= \text{Orb}_{N_1}(-E_1 + E_2) \\
&= \text{Orb}_{(F_4(-20))_{E_3, P^-}}(-E_1 + E_2). \\
(2) \quad \mathcal{P}_{P^-} &= \{E_3 + Q^+(x) + (x|x)P^- \mid x \in \mathbf{O}\} \\
&= \{\exp \mathcal{G}_1(x)E_3 \mid x \in \mathbf{O}\} \\
&= \text{Orb}_{N^+}(E_3) \\
&= \text{Orb}_{(F_4(-20))_{P^-}}(E_3). \\
(3) \quad \mathcal{P}_{Q^+(1)} &= \{E_3 + Q^+(x) + (x|x)P^- \mid x \in \text{Im}\mathbf{O}\} \\
&= \{\exp \mathcal{G}_1(x)E_3 \mid x \in \text{Im}\mathbf{O}\} \\
&= \text{Orb}_{N_2}(E_3) \\
&= \text{Orb}_{(F_4(-20))_{Q^+(1)}}(E_3).
\end{aligned}$$

Proof. (1) At first, set $\mathcal{P} = \{(-E_1 + E_2) + F_3^1(2p) + 2(p|p)P^- \mid p \in \text{Im}\mathbf{O}\}$. By Proposition 4.12(1)(i), $(-E_1 + E_2) + F_3^1(2p) + 2(p|p)P^- = \exp \mathcal{G}_2(-p)(-E_1 + E_2)$, and so $\mathcal{P} \subset \text{Orb}_{N_1}(-E_1 + E_2)$ follows.

At second, fix $X \in \mathcal{P}_{E_3, P^-}$. Because of $\mathcal{J}_{2E_3, -1}^1 = \mathbb{R}(-E_1 + E_2) \oplus \mathbb{R}P^- \oplus F_3^1(\text{Im}\mathbf{O})$, X can be expressed by $X = r(-E_1 + E_2) + F_3^1(2p) + sP^-$ for some $r, s \in \mathbb{R}$ and $p \in \text{Im}\mathbf{O}$. Then by direct calculation, $P^- \times X = -rE_3$. Therefore $r = 1$ and so $X = (-E_1 + E_2) + F_3^1(2p) + sP^-$. Next, $1 = Q_{E_3}(X) = (1 + s)^2 - s^2 - 4(p|p)$ so that $s = 2(p|p)$. Therefore $X = (-E_1 + E_2) + 2F_3^1(p) + 2(p|p)P^-$ and so $X \in \mathcal{P}$. Thus $\mathcal{P}_{E_3, P^-} \subset \mathcal{P} \subset \text{Orb}_{N_1}(-E_1 + E_2)$. Since N_1 is a subgroup of $(F_4(-20))_{E_3, P^-}$, $\text{Orb}_{N_1}(-E_1 + E_2) \subset \text{Orb}_{(F_4(-20))_{E_3, P^-}}(-E_1 + E_2)$.

At third, by virtue of the definition of \mathcal{P}_{E_3, P^-} , $(F_4(-20))_{E_3, P^-}$ acts on \mathcal{P}_{E_3, P^-} . Because of $-E_1 + E_2 \in \mathcal{P}_{E_3, P^-}$, we get $\text{Orb}_{(F_4(-20))_{E_3, P^-}}(-E_1 + E_2) \subset \mathcal{P}_{E_3, P^-}$. Consequently $\mathcal{P}_{E_3, P^-} \subset \mathcal{P} \subset \text{Orb}_{N_1}(-E_1 + E_2) \subset \text{Orb}_{(F_4(-20))_{E_3, P^-}}(-E_1 + E_2) \subset \mathcal{P}_{E_3, P^-}$. Hence (1) follows.

(2) At first, set $\mathcal{P}' = \{E_3 + Q^+(x) + (x|x)P^- \mid x \in \mathbf{O}\}$. By Proposition 4.12(2)(iv), $E_3 + Q^+(x) + (x|x)P^- = \exp \mathcal{G}_1(x)E_3$, and so $\mathcal{P}' \subset \text{Orb}_{N^+}(E_3)$ follows.

At second, fix $X \in \mathcal{P}_{P^-}$. X can be expressed by $X = r(-E_1 + E_2) + sP^- + uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y)$ for some $r, s, u, v \in \mathbb{R}$ $p \in \text{Im}\mathbf{O}$ and $x, y \in \mathbf{O}$. Then by direct calculation, $P^- \times X = -\frac{1}{2}(u + v)P^- - rE_3 + Q^+(y)$. Therefore $r = y = 0$ and $u + v = 1$, and so $X = sP^- + (1 - v)E + vE_3 + F_3^1(p) + Q^+(x)$. Next $1 = \text{tr}(X) = 3(1 - v) + v$. Therefore $v = 1$, and so $X = E_3 + sP^- + F_3^1(p) + Q^+(x)$. Because of $0 = X^{\times 2} = ((x|x) - s)P^- - F_3^1(p) + (p|p)E_3 + Q^-(px)$, we get $p = 0$ and $s = (x|x)$. Therefore $X = E_3 + (x|x)P^- + Q^+(x) \in \mathcal{P}$, and so $\mathcal{P}_{P^-} \subset \mathcal{P}' \subset \text{Orb}_{N^+}(E_3)$. Since N^+ is a subgroup of $(F_4(-20))_{P^-}$, $\text{Orb}_{N^+}(E_3) \subset \text{Orb}_{(F_4(-20))_{P^-}}(E_3)$.

At third, by virtue of the definition of \mathcal{P}_{P^-} , $(F_4(-20))_{P^-}$ acts on \mathcal{P}_{P^-} . Because of $E_3 \in \mathcal{P}_{P^-}$, $\text{Orb}_{(F_4(-20))_{P^-}}(E_3) \subset \mathcal{P}_{P^-}$. Consequently $\mathcal{P}_{P^-} \subset \mathcal{P}' \subset \text{Orb}_{N^+}(E_3) \subset \text{Orb}_{(F_4(-20))_{P^-}}(E_3) \subset \mathcal{P}_{P^-}$. Hence (2) follows.

(3) At first, set $\mathcal{P}'' = \{E_3 + Q^+(x) + (x|x)P^- \mid x \in \text{Im}\mathbf{O}\}$. Then $E_3 + Q^+(x) + (x|x)P^- = \exp \mathcal{G}_1(x)E_3$ by Proposition 4.12(2)(iv). Thus $\mathcal{P}'' \subset \text{Orb}_{N_2}(E_3)$.

At second, fix $X \in \mathcal{P}_{Q^+(1)}$. Because of $X \in \mathcal{P}_{P^-}$ and (2), X can be expressed by $X = E_3 + Q^+(x_0) + (x_0|x_0)P^-$ for some $x_0 \in \mathbf{O}$. Because of $0 = Q^+(1) \times X = \text{Re}(x_0)P^-$, we get $x_0 \in \text{Im}\mathbf{O}$. Therefore $X \in \mathcal{P}''$ and so $\mathcal{P}_{Q^+(1)} \subset \mathcal{P}''$. By Proposition 4.12, for all $x, p \in \text{Im}\mathbf{O}$, $\exp \mathcal{G}_1(x) \exp \mathcal{G}_2(p)Q^+(1) = Q^+(1) + (x|1)P^- = Q^+(1)$. Therefore N_2 is a subgroup of $(F_{4(-20)})_{Q^+(1)}$, and so $\text{Orb}_{N_2}(E_3) \subset \text{Orb}_{(F_{4(-20)})_{Q^+(1)}}(E_3)$.

At third, by virtue of the definition of $\mathcal{P}_{Q^+(1)}$, $(F_{4(-20)})_{Q^+(1)}$ acts on $\mathcal{P}_{Q^+(1)}$. Thus $\text{Orb}_{(F_{4(-20)})_{Q^+(1)}}(E_3) \subset \mathcal{P}_{Q^+(1)}$ follows from $E_3 \in \mathcal{P}_{Q^+(1)}$. Consequently $\mathcal{P}_{Q^+(1)} \subset \mathcal{P}'' \subset \text{Orb}_{N_2}(E_3) \subset \text{Orb}_{(F_{4(-20)})_{Q^+(1)}}(E_3) \subset \mathcal{P}_{Q^+(1)}$. Hence (3) follows. \square

It follows that \mathcal{P}_{E_3, P^-} , \mathcal{P}_{P^-} and $\mathcal{P}_{Q^+(1)}$ are parabolic type planes from Lemma 5.4. Let f_i be the mappings from the suitable \mathbb{R}^n to the parabolic type planes defined as

$$\begin{aligned} f_1 : \text{Im}\mathbf{O} &\rightarrow \mathcal{P}_{E_3, P^-}; & f_1(p) &:= \exp \mathcal{G}_2(p)(-E_1 + E_2) & \text{for } p \in \text{Im}\mathbf{O}, \\ f_2 : \mathbf{O} &\rightarrow \mathcal{P}_{P^-}; & f_2(x) &:= \exp \mathcal{G}_1(x)E_3 & \text{for } x \in \mathbf{O}, \\ f_3 : \text{Im}\mathbf{O} &\rightarrow \mathcal{P}_{Q^+(1)}; & f_3(x) &:= \exp \mathcal{G}_1(x)E_3 & \text{for } x \in \text{Im}\mathbf{O}. \end{aligned}$$

Lemma 5.5. f_i is a bijection for any $i \in \{1, 2, 3\}$.

Proof. Case f_1 . By Lemma 5.4(1), $\mathcal{P}_{E_3, P^-} = \{f_1(p) \mid p \in \text{Im}\mathbf{O}\}$, and so f_1 is onto. If $p, q \in \text{Im}\mathbf{O}$ and $p \neq q$, then $(-E_1 + E_2) + 2F_3^1(p) + 2(p|p)P^- \neq (-E_1 + E_2) + 2F_3^1(q) + 2(q|q)P^-$. Therefore f_1 is one-to-one.

Case f_2 . By Lemma 5.4(2), $\mathcal{P}_{P^-} = \{f_2(x) \mid x \in \mathbf{O}\}$, and so f_2 is onto. If $x, y \in \mathbf{O}$ and $x \neq y$, then $f_2(x) = E_3 + Q^+(x) + (x|x)P^- \neq E_3 + Q^+(y) + (y|y)P^- = f_2(y)$. Therefore f_2 is one-to-one.

Case f_3 . By Lemma 5.4(3), $\mathcal{P}_{Q^+(1)} = \{f_3(x) \mid x \in \text{Im}\mathbf{O}\}$, and so f_3 is onto. If $x, y \in \text{Im}\mathbf{O}$ and $x \neq y$, then $f_3(x) = E_3 + Q^+(x) + (x|x)P^- \neq E_3 + Q^+(y) + (y|y)P^- = f_3(y)$. Therefore f_3 is one-to-one. \square

The homomorphism $\varphi_1 : \text{Spin}(7) \ltimes \text{Im}\mathbf{O} \rightarrow (F_{4(-20)})_{E_3, P^-}$ is defined by the restriction $\varphi_1 := \varphi|_{\text{Spin}(7) \ltimes \text{Im}\mathbf{O}}$:

$$\varphi_1(g, p) = \varphi(g, p, 0) = \exp \mathcal{G}_2(p)\varphi_0(g) \quad \text{for } (g, p) \in \text{Spin}(7) \ltimes \text{Im}\mathbf{O}.$$

By Lemma 3.4 and Proposition 4.12(1), we get $\varphi(g, p)E_3 = E_3$ and $\varphi(g, p)P^- = P^-$, and so the mapping is well-defined. The homomorphism $\varphi_2 : G_2 \ltimes (\text{Im}\mathbf{O} \times \text{Im}\mathbf{O}) \rightarrow (F_{4(-20)})_{Q^+(1)}$ is defined by the restriction $\varphi_2 := \varphi|_{G_2 \ltimes (\text{Im}\mathbf{O} \times \text{Im}\mathbf{O})}$:

$$\varphi_2(g, p, q) = \exp(\mathcal{G}_2(p) + \mathcal{G}_1(q))\varphi_0(g) \quad \text{for } (g, p, q) \in G_2 \ltimes (\text{Im}\mathbf{O} \times \text{Im}\mathbf{O}).$$

By Lemma 3.4 and Proposition 4.12, $\varphi_2(g, p, x)Q^+(1) = Q^+(1)$, and so the mapping is well-defined.

Proposition 5.6.

- (1) φ_1 is an isomorphism onto $(F_{4(-20)})_{E_3, P^-}$.
- (2) φ is an isomorphism onto $(F_{4(-20)})_{P^-}$.
- (3) φ_2 is an isomorphism onto $(F_{4(-20)})_{Q^+(1)}$.

Proof. (1) It is enough to show φ_1 is an onto and one-to-one. Let $\tilde{g} \in (F_{4(-20)})_{E_3, P^-}$. By Lemma 5.4(1) and Proposition 4.12(1)(i),

$$\tilde{g}(-E_1 + E_2) = (-E_1 + E_2) + 2F_3^1(p) + 2(p|p)P^- = \exp \mathcal{G}_2(-p)(-E_1 + E_2)$$

for some $p \in \text{Im } \mathbf{O}$. Therefore $\exp \mathcal{G}_2(p)\tilde{g}(-E_1 + E_2) = -E_1 + E_2$. Thus by Lemmas 3.1(2) and 3.2(2), $\exp \mathcal{G}_2(p)\tilde{g} \in (F_{4(-20)})_{-E_1 + E_2, E_3, P^-} = (F_{4(-20)})_{E_1, E_3, F_3^1(1)} = \varphi_0(\text{Spin}(7))$. Therefore $\exp \mathcal{G}_2(p)\tilde{g} = \varphi_0(g)$ for some $g \in \text{Spin}(7)$, and so $\tilde{g} = \exp \mathcal{G}_2(p)\varphi_0(g) = \varphi_1(g, p)$. Hence φ_1 is onto.

Take $(g, p) \in \text{Ker}(\varphi_1)$. By Lemma 3.2(2), $-E_1 + E_2 = \varphi_1(g, p)(-E_1 + E_2) = \exp \mathcal{G}_2(p)\varphi_0(g)(-E_1 + E_2) = \exp \mathcal{G}_2(p)(-E_1 + E_2) = f_1(p)$. By Lemma 5.5, $p = 0$, so that $\varphi_1(g, p) = \varphi_0(g)$. By Lemma 3.2(2), $g = 1$. Therefore $\text{Ker}(\varphi_1) = \{(1, 0)\}$, and so φ_1 is one-to-one. Hence (1) follows.

(2) It is enough to show φ is an onto and one-to-one. Let $\tilde{g} \in (F_{4(-20)})_{P^-}$. By Lemma 5.4(2) and Proposition 4.12(2)(iv),

$$\tilde{g}E_3 = E_3 + Q^+(x) + (x|x)P^- = \exp \mathcal{G}_1(x)E_3$$

for some $x \in \mathbf{O}$. Then $\exp \mathcal{G}_1(-x)\tilde{g}P^- = P^-$ and so $\exp \mathcal{G}_1(-x)\tilde{g} \in (F_{4(-20)})_{E_3, P^-}$. By (1), $\exp \mathcal{G}_1(-x)\tilde{g} = \exp \mathcal{G}_2(p)\varphi_0(g)$ for some $(g, p) \in \text{Spin}(7) \times \text{Im } \mathbf{O}$. Therefore we obtain $\tilde{g} = \exp \mathcal{G}_1(x)\exp \mathcal{G}_2(p)\varphi_0(g) = \varphi(g, p, x)$, and so φ is onto.

Next, take $(g, p, x) \in \text{Ker}(\varphi)$. By Lemma 3.4 and Proposition 4.12(2), $\varphi_0(g), \exp \mathcal{G}_2(p) \in (F_{4(-20)})_{E_3}$. Thus $E_3 = \varphi(g, p, x)E_3 = \exp \mathcal{G}_1(x)E_3 = f_2(x)$. Then by Lemma 5.5, $x = 0$, and so $\varphi(g, p, 0) = \varphi_1(g, p) \in (F_{4(-20)})_{E_3, P}$. By (1), $(g, p) \in \text{Ker}(\varphi_1) = \{(1, 0)\}$. Therefore $\text{Ker}(\varphi) = \{(1, 0, 0)\}$ and so φ is one-to-one. Hence (2) follows.

(3) By (2), the map φ_2 is a restriction map of isomorphism φ , and so φ_2 is a mono-morphism. Therefore it is enough to show φ_2 is onto. Take $\tilde{g} \in (F_{4(-20)})_{Q^+(1)}$. Because of $P^- = Q^+(1) \times Q^+(1)$, $(F_{4(-20)})_{Q^+(1)}$ is a subgroup of $(F_{4(-20)})_{P^-}$. Therefore by (2), $\tilde{g} = \varphi(g, p, x)$ for some $(g, p, x) \in \text{Spin}(7) \times (\text{Im } \mathbf{O} \times \text{Im } \mathbf{O})$ with $g = (g_1, g_2, g_3)$. Because of Lemma 3.4 and Proposition 4.12, and $\varphi(g, p, x) \in (F_{4(-20)})_{Q^+(1)}$, $Q^+(1) = \varphi(g, p, x)Q^+(1) = Q^+(g_1 1) + 2(x|g_1 1)P^-$. Therefore $g_1 1 = 1$ and $0 = (x|g_1 1)$. Because of $(x|1) = (x|g_1 1) = 0$, we get $x \in \text{Im } \mathbf{O}$. Since $g \in \text{Spin}(7)$, $g_1 1 = 1$ and Lemma 2.4(4), we obtain $g \in G_2$. Therefore $(g, p, x) \in G_2 \times (\text{Im } \mathbf{O} \times \text{Im } \mathbf{O})$, and so φ_2 is onto. Hence (3) follows. \square

The maps $\psi_1 : F_{4(-20)} \rightarrow \mathbf{O}$, $\psi_2 : F_{4(-20)} \rightarrow \text{Im}\mathbf{O}$ and $\psi_3 : F_{4(-20)} \rightarrow F_{4(-20)}$ are defined as

$$(5.3) \quad \psi_1(g) := \{gE_3\}_{Q^+} = \frac{1}{2}((gE_3)_{F_1^1} + \overline{(gE_3)_{F_2^1}}),$$

$$(5.4) \quad \psi_2(g) := -\frac{1}{2}\{g(-E_1 + E_2)\}_{\text{Im}F_3^1} = -\frac{1}{2}\text{Im}\left((g(-E_1 + E_2))_{F_3^1}\right),$$

$$(5.5) \quad \psi_3(g) := \exp(-\mathcal{G}_1(\psi_1(g)) - \mathcal{G}_2(\psi_2(g)))g$$

for $g \in F_{4(-20)}$ (cf. Lemma 1.4), respectively.

Proposition 5.7.

(1) Let $g \in (F_{4(-20)})_{P^-}$. Then $\psi_3(g) \in M$ and

$$g = \exp(\mathcal{G}_1(\psi_1(g)) + \mathcal{G}_2(\psi_2(g)))\psi_3(g) \in N^+M.$$

(2) $(F_{4(-20)})_{P^-} = N^+M = MN^+$.

Proof. (1) By Proposition 5.6(2), g can be expressed by

$$g = \exp \mathcal{G}_2(p) \exp \mathcal{G}_1(x) \varphi_0(h)$$

for some $p \in \text{Im}\mathbf{O}$, $x \in \mathbf{O}$ and $h \in \text{Spin}(7)$. By Lemma 3.4 and Proposition 4.12(1)(2),

$$\{gE_3\}_{Q^+} = \{E_3 + Q^+(x) + (x|x)P^-\}_{Q^+} = x$$

and

$$\begin{aligned} -\frac{1}{2}\{g(-E_1 + E_2)\}_{\text{Im}F_3^1} &= -\frac{1}{2}\{(-E_1 + E_2) - 2F_3^1(p) + 2(p|p)P^- \\ &\quad - Q^-(x) + Q^+(px) - (x|x)(E - 3E_3) + (x|x)Q^+(x) \\ &\quad + \frac{1}{2}(x|x)^2P^-\}_{\text{Im}F_3^1} = p. \end{aligned}$$

Therefore $\psi_1(g) = x$ and $\psi_2(g) = p$. Then by Proposition 4.4, $\psi_3(g) = \exp(-\mathcal{G}_1(x) - \mathcal{G}_2(p))g = \varphi_0(h) \in \varphi_0(\text{Spin}(7)) = M$. Hence (1) follows.

(2) By (1) and Propositions 5.3(4), $(F_{4(-20)})_{P^-} = N^+M$. By Proposition 5.3(1), $\varphi_0(g) \exp(\mathcal{G}_2(p) + \mathcal{G}_1(x)) = \exp(\mathcal{G}_2(g_3p) + \mathcal{G}_1(g_1x))\varphi_0(g)$ for all $(g, p, x) \in \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})$ where $g = (g_1, g_2, g_3)$. Therefore $N^+M = MN^+$. Hence (2) follows. \square

Proposition 5.8. Let $p, q \in \mathbb{R}$ and $p \neq q$.

(1) For $Y = P^- + q(E - E_3) + pE_3 \in \mathcal{J}^1$,

$$(F_{4(-20)})_Y = (F_{4(-20)})_{E_3, P^-} \cong \text{Spin}(7) \ltimes \text{Im}\mathbf{O}.$$

(2) For $Y' = P^+ + q(E - E_3) + pE_3 \in \mathcal{J}^1$,

$$(F_{4(-20)})_{Y'} = \tilde{\sigma}_1((F_{4(-20)})_{E_3, P^-} \cong \text{Spin}(7) \ltimes \text{Im}\mathbf{O}.$$

Proof. (1) Let $g \in (F_{4(-20)})_{E_3, P^-}$. Then $gY = g(P^- + q(E - E_3) + pE_3) = P^- + q(E - E_3) + pE_3 = Y$. Therefore $g \in (F_{4(-20)})_Y$ and so $(F_{4(-20)})_{E_3, P^-} \subset (F_{4(-20)})_Y$.

Conversely, take $g \in (F_{4(-20)})_Y$. Then $\varphi_Y(p)^{\times 2} = (p - q)^2 E_3$ and $\text{tr}(\varphi_Y(p)^{\times 2}) = (p - q)^2 \neq 0$. Therefore $E_{Y,p} \in \mathcal{J}^1$ is well-defined and

$E_{Y,p} = E_3$. By Proposition 1.9(2), $gE_3 = gE_{Y,p} = E_{gY,p} = E_{Y,p} = E_3$, and so $gP^- = g(Y - pE_3 - q(E - E_3)) = Y - pE_3 - q(E - E_3) = P^-$. Therefore $g \in (F_{4(-20)})_{E_3,P^-}$, and so $(F_{4(-20)})_Y \subset (F_{4(-20)})_{E_3,P^-}$. Hence $(F_{4(-20)})_Y = (F_{4(-20)})_{E_3,P^-}$, and $(F_{4(-20)})_Y \cong \text{Spin}(7) \ltimes \text{Im}\mathbf{O}$ follows from Proposition 5.6(1).

(2) Obviously $(F_{4(-20)})_{Y'} = (F_{4(-20)})_{-Y'}$. Put $Z = P^- - q(E - E_3) - pE_3$. Because of $\sigma_1(-Y') = P^- - q(E - E_3) - pE_3 = Z$ and (1), $(F_{4(-20)})_{-Y'} = \sigma_1(F_{4(-20)})_Z \sigma_1 = \tilde{\sigma}_1((F_{4(-20)})_Z) = \tilde{\sigma}_1((F_{4(-20)})_{E_3,P^-})$. Therefore by Proposition 5.6(1), $(F_{4(-20)})_{Y'} = \tilde{\sigma}_1((F_{4(-20)})_{E_3,P^-}) \cong (F_{4(-20)})_{E_3,P^-} \cong \text{Spin}(7) \ltimes \text{Im}\mathbf{O}$. \square

Proposition 5.9. *Let $r \in \mathbb{R}$.*

(1) *For $Y = P^- + rE \in \mathcal{J}^1$,*

$$(F_{4(-20)})_Y = (F_{4(-20)})_{P^-} \cong \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O}).$$

(2) *For $Y' = P^+ + rE \in \mathcal{J}^1$,*

$$(F_{4(-20)})_{Y'} = \tilde{\sigma}_1((F_{4(-20)})_{P^-}) \cong \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O}).$$

Proof. (1) Since the element E is invariant under the $F_{4(-20)}$ -action, $(F_{4(-20)})_Y = (F_{4(-20)})_{P^-}$. Therefore by Proposition 5.6(2), $(F_{4(-20)})_Y \cong \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})$.

(2) $(F_{4(-20)})_{Y'} = (F_{4(-20)})_{-Y'}$ follows immediately. Put $Z = P^- - rE$. Because of $\sigma_1(-Y') = P^- - rE = Z$ and (1), $(F_{4(-20)})_{-Y'} = \tilde{\sigma}_1((F_{4(-20)})_Z) = \tilde{\sigma}_1((F_{4(-20)})_{P^-})$. Thus $(F_{4(-20)})_{Y'} = \tilde{\sigma}_1((F_{4(-20)})_{P^-}) \cong (F_{4(-20)})_{P^-} \cong \text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})$ by Proposition 5.6(2). \square

Proposition 5.10. *Let $Y = Q^+(1) + rE \in \mathcal{J}^1$ where $r \in \mathbb{R}$. Then*

$$(F_{4(-20)})_Y = (F_{4(-20)})_{Q^+(1)} \cong G_2 \ltimes (\text{Im}\mathbf{O} \times \text{Im}\mathbf{O}).$$

Proof. Since the element E is invariant under the $F_{4(-20)}$ -action, we get $(F_{4(-20)})_Y = (F_{4(-20)})_{Q^+(1)}$. Hence the assertion follows from Proposition 5.6(3). \square

Proof of Main Theorem 1. By Proposition 1.10, a concrete classification of $F_{4(-20)}$ -orbits on \mathcal{J}^1 . By Propositions 3.5, 3.11(1)(2), 3.14, 5.8, 5.9, 5.10 and 1.5, we obtain the stabilizer groups of $F_{4(-20)}$ -orbits on \mathcal{J}^1 . \square

Remark 5.11. Let \mathbf{F} denotes a real division algebra \mathbb{R} (real numbers), \mathbf{C} (complex numbers), \mathbf{H} (quaternions) or \mathbf{O} (octonions). J.A. Wolf ([42, 41]) gave *Heisenberg groups* $H_{p,q,\mathbf{F}}$ and $G_{p,q,\mathbf{F}}$. Then $H_{1,0,\mathbf{O}}$ is equal to the group $\text{Im}\mathbf{O} \times \mathbf{O}$ and $G_{1,0,\mathbf{O}}$ is equal to the group $\text{Spin}(7) \ltimes (\text{Im}\mathbf{O} \times \mathbf{O})$. F.W. Keene showed $MN^+ \cong G_{1,0,\mathbf{O}}$ in his thesis (cf. [17], [41]). In Propositions 5.6 and 5.7, it appears that the subgroup $MN^+ \cong G_{1,0,\mathbf{O}}$

of $F_{4(-20)}$ is the stabilizer of the certain element P^- in the exceptional Jordan algebra \mathcal{J}^1 .

Part II: An application of the realization of the stabilizer groups.

6. PRELIMINARIES.

Let G be a linear connected semisimple Lie group with its Lie algebra \mathfrak{g} over \mathbb{R} . Let θ be a Cartan involution of \mathfrak{g} , $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition, \mathfrak{a} a maximal abelian subspace of \mathfrak{p} , $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. \mathfrak{a}^* denotes the dual space of \mathfrak{a} . For any element $\lambda \in \mathfrak{a}^*$, let $\mathfrak{g}_{\lambda} := \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$. λ is called a *root* of $(\mathfrak{g}, \mathfrak{a})$ if $\lambda \neq 0$ and $\mathfrak{g}_{\lambda} \neq \{0\}$. The set of roots of $(\mathfrak{g}, \mathfrak{a})$ is denoted by Σ . Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_{\lambda}$ follows. Let us denote by Σ^+ a set of positive root of $(\mathfrak{g}, \mathfrak{a})$ with respect to the some ordering in \mathfrak{a}^* , $\Sigma^- := \{-\lambda \mid \lambda \in \Sigma^+\}$, $\mathfrak{n}^+ := \sum_{\lambda \in \Sigma^+} \mathfrak{g}_{\lambda}$ and $\mathfrak{n}^- := \sum_{\lambda \in \Sigma^-} \mathfrak{g}_{\lambda}$. Then \mathfrak{n}^+ and \mathfrak{n}^- are nilpotent subalgebras, $\theta \mathfrak{n}^{\pm} = \mathfrak{n}^{\mp}$ (*resp*), and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}^+$ follow (cf. [19, Ch V]). Suppose that there exists an involutive automorphism Θ on G such that the differential $d\Theta = \theta$, and the center $Z(G)$ of G is finite. Let us denote the subgroup $K := G^{\Theta}$ of G . Then $\text{Lie}(K) = \mathfrak{k}$ and K is connected, closed, and K is a maximal compact subgroup of G (cf. [15, Ch VI, Theorem 1.1]). Set $A := \exp \mathfrak{a}$, $M := Z_K(\mathfrak{a}) = \{k \in K \mid kXk^{-1} = X \text{ for all } X \in \mathfrak{a}\}$ and $N^{\pm} := \exp \mathfrak{n}^{\pm}$ (*resp*). Then the identity connected component M^0 of M is the analytic subgroup corresponding to \mathfrak{m} , and $\Theta N^{\pm} = N^{\mp}$ (*resp*). Let us denote the normalizer of \mathfrak{a} of the group K as $M^* := N_K(\mathfrak{a}) = \{k \in K \mid k\mathfrak{a}k^{-1} \subset \mathfrak{a}\}$ and the finite factor group $W := M^*/M$. For all $w \in W$, we fix a representative $\bar{w} \in M^*$. Then the following decompositions:

$$(1) \ G = \coprod_{w \in W} MAN^+ \bar{w} N^- \quad (\text{Bruhat decomposition}),$$

$$(1)' \ G = \overline{MAN^+ N^-} \quad (\text{Gauss decomposition}),$$

$$(2) \ G = KAN^+ \quad (\text{Iwasawa decomposition}).$$

(cf. [15], [18], [26], [23]). In (1)', the set $MAN^+ N^-$ is open dense in G , and so almost any $g \in G$ can be expressed by

$$g = m_G(g) a_G(g) n_G(g) \bar{n}_G(g)$$

for some $m_G(g) \in M$, $a_G(g) \in A$, $n_G(g) \in N^+$ and $\bar{n}_G(g) \in N^-$ with uniquely determined factors. In (2), any $g \in G$ can be uniquely expressed by

$$g = k(g)(\exp H(g))n(g)$$

for some $k(g) \in K, H(g) \in \mathfrak{a}$ and $n(g) \in N$. A *signature of roots* is defined by the mapping ϵ of Σ to $\{-1, 1\}$ such that ϵ satisfies the conditions:

- (i) $\epsilon(\lambda) = \epsilon(-\lambda)$ for all $\lambda \in \Sigma$,
- (ii) $\epsilon(\lambda + \mu) = \epsilon(\lambda)\epsilon(\mu)$ if $\lambda, \mu, \lambda + \mu \in \Sigma$

[26, Definition 1.1]. For the Cartan involution θ and any signature ϵ of roots, an involutive automorphism θ_ϵ of \mathfrak{g} is defined as

- (i) $\theta_\epsilon(X) := \epsilon(\lambda)\theta(X)$ for all $\lambda \in \Sigma$ and $X \in \mathfrak{g}_\lambda$,
- (ii) $\theta_\epsilon(X) := \theta(X)$ for all $X \in \mathfrak{a} \oplus \mathfrak{m}$

[26, Definition 1.2]. θ_ϵ is called the (θ, ϵ) -*involution* of \mathfrak{g} . Set

$$\mathfrak{k}_\epsilon := \{X \in \mathfrak{g} \mid \theta_\epsilon X = X\} \text{ and } \mathfrak{p}_\epsilon := \{X \in \mathfrak{g} \mid \theta_\epsilon X = -X\}.$$

Then $\mathfrak{g} = \mathfrak{k}_\epsilon \oplus \mathfrak{p}_\epsilon$. Let $(K_\epsilon)^0$ be the analytic subgroup of G with the Lie algebra \mathfrak{k}_ϵ . The subgroup K_ϵ of G is defined by

$$K_\epsilon := (K_\epsilon)^0 M.$$

In fact, since all elements of M normalize $(K_\epsilon)^0$ by [26, Lemma 1.4(i)], K_ϵ is a subgroup of G . Let us denote

$$M_\epsilon^* := K_\epsilon \cap M^*, \quad W_\epsilon := M_\epsilon^* / M.$$

Proposition 6.1. ([26, Proposition 1.10(Iwasawa decomposition with respect to K_ϵ in the sense of T. Oshima and J. Sekiguchi)]) *Let the factor set $W_\epsilon \setminus W = \{w_1 = 1, w_2, \dots, w_r\}$ where $r = [W : W_\epsilon]$. Fix representatives $\bar{w}_1 = 1, \bar{w}_2, \dots, \bar{w}_r \in M_\epsilon^* = K_\epsilon \cap M^*$ for $w_1 = 1, w_2, \dots, w_r$. Then the decomposition*

$$G \supset \cup_{i=1}^r K_\epsilon \bar{w}_i A N^+$$

has the following properties.

- (1) *If $k\bar{w}_i a n = k'\bar{w}_j a' n'$ with $k, k' \in K_\epsilon$, $a, a' \in A$ and $n, n' \in N^+$, then $k = k'$, $i = j$, $a = a'$ and $n = n'$.*
- (2) *The map $(k, a, n) \mapsto k\bar{w}_i a n$ defines an analytic diffeomorphism of the product manifold $K_\epsilon \times A \times N^+$ onto the open submanifold $K_\epsilon \bar{w}_i A N^+$ of G ($i = 1, \dots, r$).*
- (3) *The submanifolds $\cup_{i=1}^r K_\epsilon \bar{w}_i A N^+$ is open dense in G .*

$F_{4(-20)}$ is a linear connected semisimple Lie group and $Z(F_{4(-20)}) = \{1\}$ (cf. [40, Theorem 2.14.1, Theorem 2.14.2]). We recall that the differential $d\tilde{\sigma}_1 = \tilde{\sigma}_1$ of the involutive automorphism $\tilde{\sigma}_1$ on $F_{4(-20)}$ is a Cartan involution by Lemma 4.2(3), $\mathfrak{f}_{4(-20)} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, and \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} by Lemma 4.5(2). We note $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, $\Sigma = \{\pm\alpha, \pm 2\alpha\}$, $\mathfrak{n}^\pm = \mathfrak{g}_{\pm\alpha} \oplus \mathfrak{g}_{\pm 2\alpha}$, $A = \exp \mathfrak{a}$, $N^\pm = \exp \mathfrak{n}^\pm$, and $\tilde{\sigma}_1(N^\pm) = N^\mp$ (resp). By Proposition 3.16(2), $K = (F_{4(-20)})^{\tilde{\sigma}_1} = \text{Spin}(9)$, and by Proposition 4.4, $M = \varphi_0(\text{Spin}(7))$.

So K and M are analytic subgroups of $F_{4(-20)}$ with their Lie algebras \mathfrak{k} and \mathfrak{m} , respectively. Let ϵ be a signature of root defined by

$$\epsilon(\alpha) = \epsilon(-\alpha) := -1, \quad \epsilon(2\alpha) = \epsilon(-2\alpha) := 1.$$

Let us denote the $(\tilde{\sigma}_1, \epsilon)$ -involution by $(\tilde{\sigma}_1)_\epsilon$, and use same notations $\mathfrak{k}_\epsilon, (K_\epsilon)^0, K_\epsilon, M^*, M_\epsilon^*, W$ and W_ϵ corresponding to notations of general G , respectively.

Lemma 6.2. *The following assertion hold.*

- (1) $(\tilde{\sigma}_1)_\epsilon = \tilde{\sigma}_2$.
- (2) $K_\epsilon = (F_{4(-20)})_{E_2}$.
- (3) $M^* = M \amalg \sigma_1 M$. Especially, $W = \{M, \sigma_1 M\} \cong \mathbb{Z}_2$.
- (4) $M_\epsilon^* = M \amalg \sigma_1 M$. Especially, $W_\epsilon = \{M, \sigma_1 M\}$ and $[W : W_\epsilon] = 1$.

Proof. (1) By Proposition 4.4, $\mathfrak{m} \subset \mathfrak{d}_4$, and so by Lemma 4.1(1), $\tilde{\sigma}_2(t\tilde{A}_3^1(1) + D) = -t\tilde{A}_3^1(1) + D = \tilde{\sigma}_1(t\tilde{A}_3^1(1) + D)$ for all $t \in \mathbb{R}$ and $D \in \mathfrak{m}$. Thus the condition (ii) of $(\tilde{\sigma}_1)_\epsilon$ follows. For all $x \in \mathbf{O}$ and $p \in \text{Im}\mathbf{O}$, because of Lemma 4.1(1) and direct calculation, we get $\tilde{\sigma}_2\mathcal{G}_{\pm 1}(x) = -\mathcal{G}_{\mp 1}(x) = \epsilon(\pm\alpha)\tilde{\sigma}_1\mathcal{G}_{\pm 1}(x)$ and $\tilde{\sigma}_2\mathcal{G}_{\pm 2}(p) = \mathcal{G}_{\mp 2}(p) = \epsilon(\pm 2\alpha)\tilde{\sigma}_1\mathcal{G}_{\pm 2}(p)$, respectively. Therefore $\tilde{\sigma}_2\phi = \epsilon(\lambda)\tilde{\sigma}_1\phi$ for all $\lambda \in \Sigma$ and $\phi \in \mathfrak{g}_\lambda$, and so the condition (i) of $(\tilde{\sigma}_1)_\epsilon$ follows. Hence (1) follows.

(2) By (1), $\mathfrak{k}_\epsilon = \{\phi \in \mathfrak{f}_{4(-20)} \mid \tilde{\sigma}_2\phi = \phi\} = \text{Lie}((F_{4(-20)})^{\tilde{\sigma}_2})$. By Proposition 3.16(3), $(F_{4(-20)})_{E_2}$ is the analytic subgroup of $F_{4(-20)}$ with the Lie algebra \mathfrak{k}_ϵ . By Propositions 4.4 and 3.16(3), $M = B_3 \subset (F_{4(-20)})_{E_2}$. Thus $K_\epsilon = (K_\epsilon)^0 M = (F_{4(-20)})_{E_2} M = (F_{4(-20)})_{E_2}$. Hence (2) follows.

(3) Take $g \in M$. By Lemma 4.1(1), $\sigma_1 g \tilde{A}_3^1(1) g^{-1} \sigma_1 = \tilde{\sigma}_1 \tilde{A}_3^1(1) = \tilde{A}_3^1(-1)$. Therefore $\sigma_1 g \in M^*$ and $\sigma_1 g \notin M$, and so $\sigma_1 M \subset M^*$ and $M \cap \sigma_1 M = \emptyset$. Therefore $M \cup \sigma_1 M = M \amalg \sigma_1 M \subset M^*$.

Conversely, take $k \in M^*$. Then $k\tilde{A}_3^1(1)k^{-1} = \tilde{A}_3^1(t)$ for some $t \in \mathbb{R}$. We consider the equation $k\tilde{A}_3^1(1)k^{-1}E_1 = \tilde{A}_3^1(t)E_1$. By Proposition 3.16(2) and Lemma 3.6(2), the right hand side is $k\tilde{A}_3^1(1)k^{-1}E_1 = k\tilde{A}_3^1(1)E_1 = -kF_3^1(1)$ and the left hand side is $\tilde{A}_3^1(t)E_1 = -F_3^1(t)$. Thus $kF_3^1(1) = F_3^1(t)$. Then $-2t^2 = (F_3^1(t)|F_3^1(t)) = (kF_3^1(1)|kF_3^1(1)) = (F_3^1(1)|F_3^1(1)) = -2$. Thus $t = \pm 1$ and so $kF_3^1(1) = F_3^1(\pm 1)$. Now, if $kF_3^1(1) = F_3^1(1)$ then put $g_0 = k \in K = (F_{4(-20)})_{E_1}$, and if $kF_3^1(1) = F_3^1(-1)$ then put $g_0 = \sigma_1 k \in K = (F_{4(-20)})_{E_1}$. Therefore $g_0 F_3^1(1) = F_3^1(1)$, and so $g_0 \in (F_{4(-20)})_{E_1, F_3^1(1)} = B_3 = M$ by Lemma 3.1(2) and Proposition 4.4. It implies that $k \in M \amalg \sigma_1 M$. Thus $M^* \subset M \amalg \sigma_1 M$. Hence $M^* = M \amalg \sigma_1 M$ follows.

(4) Since $\sigma_1 E_2 = E_2$ and (2), $\sigma_1 \in (F_{4(-20)})_{E_2} = K_\epsilon$. Therefore $\sigma_1 \in K_\epsilon \cap M^* = M_\epsilon^*$. Since (3) and M is a subgroup of M_ϵ^* , $M^* = M \amalg \sigma_1 M \subset M_\epsilon^* \subset M^*$. Hence (4) follows. \square

Let us denote $\mathcal{J}^1(2; \mathbf{K}) := \{\xi_1 E_1 + \xi_2 E_2 + F_3^1(x) \mid \xi_i \in \mathbb{R}, x \in \mathbf{K}\}$ where $\mathbf{K} = \mathbf{O}$ or \mathbb{R} . By direct calculation, we have:

Lemma 6.3.

- (1) $\mathcal{J}^1 = \mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O}) \oplus Q^-(\mathbf{O})$.
- (2) $\mathcal{J}^1(2; \mathbf{O}) = \mathbb{R}(-E_1 + E_2) \oplus \mathbb{R}P^- \oplus \mathbb{R}(E - E_3) \oplus F_3^1(\text{Im}\mathbf{O})$.
- (3) $\mathcal{J}^1(2; \mathbb{R}) = \mathbb{R}(-E_1 + E_2) \oplus \mathbb{R}P^- \oplus \mathbb{R}(E - E_3)$.

Lemma 6.4.

- (1) Assume that $X \in \mathcal{J}^1$ and $(P^-|X) \neq 0$. Let

$$n_1 = \exp \mathcal{G}_1 \left(\frac{(X)_{F_1^1} - \overline{(X)_{F_2^1}}}{(P^-|X)} \right) \in N^+.$$

Then

- (i) $(P^-|n_1X) = (P^-|X)$,
- (ii) $n_1X \in \mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})$,
- (iii) $\text{Im}((n_1X)_{F_3^1}) = \text{Im}((X)_{F_3^1})$.

- (2) Assume that $X \in \mathcal{J}^1(2; \mathbf{O})$ and $(P^-|X) \neq 0$. Let

$$n_2 = \exp \mathcal{G}_2 \left(\frac{\text{Im}((X)_{F_3^1})}{(P^-|X)} \right) \in N^+.$$

Then

$$(P^-|n_2X) = (P^-|X) \text{ and } n_2X \in \mathcal{J}^1(2; \mathbb{R}).$$

- (3) Let $a_t = \exp(t\tilde{A}_3^1(1)) \in A$ ($t \in \mathbb{R}$). Then for all $r, s, u \in \mathbb{R}$,

$$\begin{aligned} & a_t(r(-E_1 + E_2) + sP^- + u(E - E_3)) \\ &= re^{-2t}(-E_1 + E_2) + (r \sinh 2t + se^{2t})P^- + u(E - E_3) \end{aligned}$$

Furthermore, $a_t\mathcal{J}^1(2; \mathbb{R}) \subset \mathcal{J}^1(2; \mathbb{R})$. Especially,

$$a_{-\frac{1}{2}\log s}(sP^-) = P^- \quad \text{for all } s > 0.$$

- (4) Let $p \in \text{Im}\mathbf{O}$ and $x, y \in \mathbf{O}$. Then

$$\left\{ \begin{array}{ll} \text{(i)} & \sigma_1(-E_1 + E_2) = -E_1 + E_2, \\ \text{(ii)} & \sigma_1P^- = 2(-E_1 + E_2) - P^-, \\ \text{(iii)} & \sigma_1F_3^1(p) = -F_3^1(p), \quad \text{(iv)} \quad \sigma_1E = E, \quad \text{(v)} \quad \sigma_1E_3 = E_3, \\ \text{(vi)} & \sigma_1Q^+(x) = Q^-(x), \quad \text{(vii)} \quad \sigma_1Q^-(y) = Q^+(y). \end{array} \right.$$

Proof. (1) At first, X can be expressed by

$$X = r(-E_1 + E_2) + sP^- + uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y)$$

for some $r, s, u, v \in \mathbb{R}$, $p \in \text{Im}\mathbf{O}$ and $x, y \in \mathbf{O}$. By Lemma 1.4, $r = \frac{1}{2}(P^-|X)$, $\frac{1}{r}y = \frac{1}{(P^-|X)}((X)_{F_1^1} - \overline{(X)_{F_2^1}})$. In the equations of Proposition 4.12(2), we note that the equations (i) has the terms of $-E_1 + E_2$ and other equations have not the terms of $-E_1 + E_2$, and the equations (i) and (vii) have the terms of $Q^-(\cdot)$ and other equations have not the terms of $Q^-(\cdot)$, moreover, the equations (v) and (vii) have the terms

of $F_3^1(\cdot)$ and other equations have not the terms of $F_3^1(\cdot)$. Therefore by Proposition 4.12(2)(i) and Lemma 1.4(i),

$$(P^-|n_1X) = 2\{n_1X\}_{-E_1+E_2} = 2r = 2\{X\}_{-E_1+E_2} = (P^-|X).$$

Hence (i) follows. At second, since Proposition 4.12(2) and $n_1 = \exp \mathcal{G}_1(\frac{1}{r}y)$,

$$\begin{aligned} & \{n_1(r(-E_1 + E_2) + sP^- + uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y))\}_{Q^-} \\ &= \{n_1(r(-E_1 + E_2))\}_{Q^-} + \{n_1Q^-(y)\}_{Q^-} + \{n_1(\text{other terms})\}_{Q^-} \\ &= -Q^-(y) + Q^-(y) + 0 = 0 \end{aligned}$$

Therefore $\{n_1X\}_{Q^-} = 0$ and so $n_1X \in \mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})$. Hence (ii) follows. At last, by Proposition 4.12(2) and Lemma 1.4(ii),

$$\begin{aligned} \text{Im}((n_1X)_{F_3^1}) &= \{n_1X\}_{\text{Im}F_3^1} \\ &= \{n_1(F_3^1(q) + Q^-(y) + (\text{other terms}))\}_{\text{Im}F_3^1} \\ &= \{n_1F_3^1(q)\}_{\text{Im}F_3^1} + \{n_1Q^-(y)\}_{\text{Im}F_3^1} + \{n_1(\text{other terms})\}_{\text{Im}F_3^1} \\ &= q + 2\text{Im}\left(\frac{y}{r} \bar{y}\right) + 0 = q = \{X\}_{\text{Im}F_3^1} = \text{Im}((X)_{F_3^1}). \end{aligned}$$

Hence (iii) follows.

(2) By Lemma 6.3(2), X can be expressed by

$$X = r(-E_1 + E_2) + sP^- + u(E - E_3) + F_3^1(p)$$

for some $r, s, u \in \mathbb{R}$ and $p \in \text{Im}\mathbf{O}$. By Lemma 1.4(i)(ii), $r = \frac{1}{2}(P^-|X)$ and $\frac{1}{2r}p = \frac{1}{(P^-|X)}\text{Im}((X)_{F_3^1})$. Since $n_2 = \exp \mathcal{G}_2(\frac{1}{2r}p)$ and Proposition 4.12(1),

$$\begin{aligned} n_2X &= r((-E_1 + E_2) - \frac{1}{r}F_3^1(p) + \frac{1}{2r^2}(p|p)P^-) + sP^- + u(E - E_3) \\ &\quad + (F_3^1(p) - \frac{1}{r}(p|p)P^-) \\ &= r(-E_1 + E_2) + (s - \frac{1}{2r}(p|p))P^- + u(E - E_3). \end{aligned}$$

Therefore by Lemma 1.4(i),

$$(P^-|n_2X) = 2\{n_2X\}_{-E_1+E_2} = 2r = 2\{X\}_{-E_1+E_2} = (P^-|X).$$

Moreover $\{n_2X\}_{\text{Im}F_3^1} = 0$ and so $n_2X \in \mathcal{J}^1(2; \mathbb{R})$. Hence (2) follows.

(3) Since Lemma 3.7(2) and direct calculation,

$$\begin{cases} a_t(-E_1 + E_2) = e^{-2t}(-E_1 + E_2) + \sinh 2tP^-, \\ a_tP^- = e^{2t}P^-, \quad a_t(E - E_3) = E - E_3. \end{cases}$$

Hence (3) follows from Lemma 6.3(3).

(4) It follows from direct calculation. □

7. THE BRUHAT AND GAUSS DECOMPOSITION OF $F_{4(-20)}$.

By Propositions 1.8(2) and 5.7(2),

$$\mathcal{N}_1^-(\mathbf{O}) \simeq F_{4(-20)}/N^+M$$

so we consider AN^- -orbits on $\mathcal{N}_1^-(\mathbf{O})$ to give the Bruhat and Gauss decomposition. For all $X \in \mathcal{N}_1^-(\mathbf{O})$, let us denote

$$\begin{aligned} a_X^G &:= \exp\left(-\frac{1}{2} \log \left(\frac{(P^-|\sigma_1 X)}{4} \right) \tilde{A}_3^1(1) \right) \in A, \\ \bar{n}_X^G &:= \tilde{\sigma}_1 \left(\exp \left(\mathcal{G}_1 \left(\frac{(\sigma_1 X)_{F_1^1} - \overline{(\sigma_1 X)_{F_2^1}}}{(P^-|\sigma_1 X)} \right) \right. \right. \\ &\quad \left. \left. + \mathcal{G}_2 \left(\frac{\text{Im}((\sigma_1 X)_{F_3^1})}{(P^-|\sigma_1 X)} \right) \right) \right) \in N^-. \end{aligned}$$

Lemma 7.1.

- (1) $(\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{N}_1^-(\mathbf{O}) = \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{N}_1^-(\mathbf{O})$.
- (2) $\mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{N}_1^-(\mathbf{O}) = \{r(-E_1 + E_2 - \frac{1}{2}P^-) \mid r > 0\} \coprod \{sP^- \mid s > 0\}$.

Proof. (1) Obviously, $\mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{N}_1^-(\mathbf{O}) \subset (\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{N}_1^-(\mathbf{O})$. Take $X \in (\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{N}_1^-(\mathbf{O})$. Then X can be expressed by

$$X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_3^1(y) + Q^+(x)$$

for some $\xi_i \in \mathbb{R}$ and $x, y \in \mathbf{O}$. Since $X \in \mathcal{N}_1^-(\mathbf{O})$ and Lemma 1.3(2),

- (i) $\xi_1 = (X|E_1) < 0$, (ii) $0 = \text{tr}(X) = \xi_1 + \xi_2 + \xi_3$,
- (iii) $0 = (X^{\times 2})_{E_1} = \xi_2 \xi_3 - (x|x)$,
- (iv) $0 = (X^{\times 2})_{E_2} = \xi_3 \xi_1 + (x|x)$.

By (iii)+(iv), $\xi_3(\xi_1 + \xi_2) = 0$. By (ii), $(\xi_1 + \xi_2) + \xi_3 = 0$ and $\xi_3(\xi_1 + \xi_2) = 0$. Therefore $\xi_3 = 0$ and $\xi_1 + \xi_2 = 0$. By (iv), $0 = (X_1 \times X_1)_{E_2} = (x|x)$ iff $x = 0$. Therefore $X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_3^1(y) \in \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{N}_1^-(\mathbf{O})$, and so $(\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{N}_1^-(\mathbf{O}) \subset \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{N}_1^-(\mathbf{O})$. Hence $(\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{N}_1^-(\mathbf{O}) = \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{N}_1^-(\mathbf{O})$.

(2) Put $\mathcal{P} = \{r(-E_1 + E_2 - \frac{1}{2}P^-) \mid r > 0\} \coprod \{sP^- \mid s > 0\}$. For $r > 0$ and $s > 0$, $r(-E_1 + E_2 - \frac{1}{2}P^-), sP^- \in \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{N}_1^-(\mathbf{O})$ follow from direct calculations. Therefore $\mathcal{P} \subset \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{N}_1^-(\mathbf{O})$. Conversely, take $X \in \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{N}_1^-(\mathbf{O})$. By Lemma 6.3(3),

$$X = r(-E_1 + E_2) + sP^- + u(E - E_3)$$

for some $r, s, u \in \mathbb{R}$. Since $X \in \mathcal{N}_1^-(\mathbf{O})$,

- (i) $-(r + s) = (X|E_1) < 0$, (ii) $0 = \text{tr}(X) = 2u$,
- (iii) $0 = (X^{\times 2})_{E_3} = -r^2 - 2rs + u^2$.

By (ii), $u = 0$ and so by (iii), $s = -\frac{r}{2}$ or $r = 0$.

Case $s = -\frac{r}{2}$. Then $X = r(-E_1 + E_2 - \frac{1}{2}P^-)$, and $r > 0$ by (i).

Case $r = 0$. Then $X = sP^-$, and so $s > 0$ by (i).

Therefore $X \in \mathcal{P}$ and so $\mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{N}_1^-(\mathbf{O}) \subset \mathcal{P}$. Hence $\mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{N}_1^-(\mathbf{O}) = \mathcal{P}$. \square

Lemma 7.2. *Let $X \in \mathcal{N}_1^-(\mathbf{O})$ where $(P^-|\sigma_1 X) \neq 0$.*

- (1) *The element $\bar{n}_X^G \in N^+$ is well-defined.*
- (2) *$(P^-|\sigma_1 X) > 0$. Especially, the element $a_X^G \in A$ is well-defined.*
- (3) *$a_X^G \bar{n}_X^G X = P^-$. Especially, $X \in \text{Orb}_{AN^-}(P^-)$.*

Proof. (1) It follows from $(P^-|\sigma_1 X) \neq 0$.

(2)(3) By Proposition 1.8(2), recall that $F_{4(-20)}$ acts on $\mathcal{N}_1^-(\mathbf{O})$. Put $n_1 = \exp \mathcal{G}_1 \left(\frac{(\sigma_1 X)_{F_1^1} - (\sigma_1 X)_{F_2^1}}{(P^-|\sigma_1 X)} \right) \in N^+$. By Lemma 6.4(1),

- (i) $(P^-|n_1 \sigma_1 X) = (P^-|\sigma_1 X)$,
- (ii) $n_1 \sigma_1 X \in \mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})$,
- (iii) $\text{Im}((n_1 \sigma_1 X)_{F_3^1}) = \text{Im}((\sigma_1 X)_{F_3^1})$.

Since $n_1 \sigma_1 X \in \mathcal{N}_1^-(\mathbf{O})$ and (ii), $n_1 \sigma_1 X \in (\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{N}_1^-(\mathbf{O})$. Applying Lemma 7.1(1),

$$n_1 \sigma_1 X \in \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{N}_1^-(\mathbf{O}).$$

Put $n_2 = \exp \mathcal{G}_2 \left(\frac{\text{Im}((n_1 \sigma_1 X)_{F_3^1})}{(P^-|n_1 \sigma_1 X)} \right) \in N^+$. Using Lemma 6.4(2) and $n_2 n_1 \sigma_1 X \in \mathcal{N}_1^-(\mathbf{O})$,

$$n_2 n_1 \sigma_1 X \in \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{N}_1^-(\mathbf{O}),$$

and then $(P^-|n_2 n_1 \sigma_1 X) = (P^-|n_1 \sigma_1 X) = (P^-|\sigma_1 X)$. Therefore by Lemma 7.1(2),

$$n_2 n_1 \sigma_1 X \in \{r(-E_1 + E_2 - \frac{1}{2}P^-) \mid r > 0\} \coprod \{sP^- \mid s > 0\}.$$

Now $(P^-| -E_1 + E_2 - \frac{1}{2}P^-) = 2$ and $(P^-|sP^-) = 0$. Thus since $(P^-|n_2 n_1 \sigma_1 X) = (P^-|\sigma_1 X) \neq 0$, we obtain $n_2 n_1 \sigma_1 X \in \{r(-E_1 + E_2 - \frac{1}{2}P^-) \mid r > 0\}$. Then $n_2 n_1 \sigma_1 X$ can be expressed by

$$n_2 n_1 \sigma_1 X = r(-E_1 + E_2 - \frac{1}{2}P^-) \text{ for some } r > 0.$$

Therefore

$$(P^-|\sigma_1 X) = (P^-|n_2 n_1 \sigma_1 X) = (P^-|r(-E_1 + E_2 - \frac{1}{2}P^-)) = 2r > 0,$$

and so (2) follows. Now by (i) and (iii), $n_2 = \exp \mathcal{G}_2 \left(\frac{\text{Im}((\sigma_1 X)_{F_3^1})}{(P^-|\sigma_1 X)} \right)$, and so by Lemma 4.6(3)(1), we see that $\sigma_1 n_2 n_1 \sigma_1 = \bar{n}_X^G \in N^-$. Then

$$\bar{n}_X^G X = \sigma_1 n_2 n_1 \sigma_1 X = \sigma_1 r(-E_1 + E_2 - \frac{1}{2}P^-) = \frac{r}{2}P^- = \frac{(P^-|\sigma_1 X)}{4}P^-$$

by Lemma 6.4(4). Therefore by Lemmas 6.4(3), $a_X^G \bar{n}_X^G X = P^-$. Hence (3) follows. \square

Lemma 7.3.

(1) Assume that $X \in \mathcal{N}_1^-(\mathbf{O})$ and $(P^-|X) = 0$. Then

$$X = -(X|E_1)P^- \text{ and } (X|E_1) < 0.$$

(2) Assume that $X \in \mathcal{N}_1^-(\mathbf{O})$ and $(P^-|\sigma_1 X) = 0$. Then

$$(i) \ X = -(X|E_1)\sigma_1 P^- \text{ and } (X|E_1) < 0,$$

$$(ii) \ N^- \subset (F_{4(-20)})_X,$$

$$(iii) \ \exp\left(\frac{1}{2} \log(-(X|E_1))\right) \tilde{A}_3^1(1)X = \sigma_1 P^-,$$

$$(iv) \ X \in Orb_A(\sigma_1 P^-) = Orb_{AN^-}(\sigma_1 P^-).$$

Proof. (1) By Lemma 1.4(i), $\{X\}_{-E_1+E_2} = 0$.

(Step 1) We will show $\{X\}_{P^-} \neq 0$. Suppose that $\{X\}_{P^-} = 0$. Then X can be expressed by

$$X = uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y)$$

for some $u, v \in \mathbb{R}$, $p \in \text{Im } \mathbf{O}$ and $x, y \in \mathbf{O}$. Since $X \in \mathcal{N}_1^-(\mathbf{O})$,

$$(i) \ u = (X|E_1) < 0, \ (ii) \ 0 = (X^{\times 2})_{E_3} = u^2 + (p|p).$$

By (ii), $u = 0$, and so it contradicts with (i). Therefore $\{X\}_{P^-} \neq 0$.

(Step 2) We will show $\{X\}_E = \{X\}_{E_3} = \{X\}_{\text{Im } F_3^1} = 0$. X can be expressed by

$$X = sP^- + uE + vE_3 + F_3^1(p) + Q^+(x) + Q^-(y)$$

for some $(0 \neq) s \in \mathbb{R}$, $u, v \in \mathbb{R}$, $p \in \text{Im } \mathbf{O}$ and $x, y \in \mathbf{O}$. Since $X \in \mathcal{N}_1^-(\mathbf{O})$,

$$(i) \ 0 = (X^{\times 2})_{E_3} = u^2 + (p|p), \ (ii) \ 0 = \text{tr}(X) = 3u + v.$$

By (i), $u = p = 0$, and so $v = 0$ by (ii).

(Step 3) We will show $X = sP^-$ for some $s > 0$. By (Step 1) and (Step 2), X can be expressed by

$$X = sP^- + Q^+(x) + Q^-(y) = sP^- + F_1^1(z) + F_2^1(w)$$

for some $(0 \neq) s \in \mathbb{R}$ and $z, w \in \mathbf{O}$. Since $X \in \mathcal{N}_1^-(\mathbf{O})$, $0 = (X^{\times 2})_{E_1} = -(z|z)$ and $0 = (X^{\times 2})_{E_2} = (w|w)$. Therefore $z = w = 0$, so that $X = sP^-$. Then $s = -(X|E_1)$, and because of $X \in \mathcal{N}_1^-(\mathbf{O})$, $(X|E_1) < 0$. Hence (1) follows.

(2) At first, $\sigma_1 P^- \in Orb_{F_{4(-20)}}(P^-) = \mathcal{N}_1^-(\mathcal{O})$. Thus by (1), $\sigma_1 X = -(\sigma_1 X|E_1)P^-$ and $(\sigma_1 X|E_1) < 0$. Now $(\sigma_1 X|E_1) = (X|\sigma_1 E_1) = (X|E_1)$. Therefore $X = -(X|E_1)\sigma_1 P^-$, and $(X|E_1) < 0$. Hence (i) follows. At second, by Proposition 5.7(2), $N^+ \subset (F_{4(-20)})_{P^-}$. Thus by Lemma 4.6(3) and (i), we have

$$N^- = \tilde{\sigma}_1(N^+) \subset \tilde{\sigma}_1((F_{4(-20)})_{P^-}) = (F_{4(-20)})_{\sigma_1 P^-} = (F_{4(-20)})_X.$$

Hence (ii) follows. At third, put $a_t = \exp(t\tilde{A}_3^1(1)) \in A$ with $t = \frac{1}{2} \log(-(X|E_1))$. Using (i) and Lemma 6.4(4)(3),

$$\begin{aligned} a_t X &= -(X|E_1)a_t \sigma_1 P^- = -(X|E_1)a_t(2(-E_1 + E_2) - P^-) \\ &= -(X|E_1)e^{-2t}(2(-E_1 + E_2) - P^-) \\ &= 2(-E_1 + E_2) - P^- = \sigma_1 P^-. \end{aligned}$$

Hence (iii) follows. At last, by (iii), $X \in \text{Orb}_A(\sigma_1 P^-)$ and so by (ii), $X \in \text{Orb}_A(\sigma_1 P^-) = \text{Orb}_{AN^-}(\sigma_1 P^-)$. Hence (iv) follows. \square

By Lemmas 7.2(3) and 7.3(2), we have:

Corollary 7.4. $\mathcal{N}_1^-(\mathbf{O}) = \text{Orb}_{AN^-}(P^-) \amalg \text{Orb}_{AN^-}(\sigma_1 P^-)$.

Lemma 7.5. Let $m \in M$, $a_t = \exp(t\tilde{A}_3^1(1)) \in A$ ($t \in \mathbb{R}$), $n \in N^+$, $\bar{n} \in N^-$, $x \in \mathbf{O}$ and $p \in \text{Im}\mathbf{O}$.

- (1) (i) $ma_t n P^- = e^{2t} P^-$, (ii) $(ma_t n)^{-1} P^- = e^{-2t} P^-$.
- (2) $a_t \exp(\mathcal{G}_1(x) + \mathcal{G}_1(p))a_t^{-1} = \exp(t(\mathcal{G}_1(x) + 2\mathcal{G}_1(p)))$.
- (3) $\sigma_1 \exp \mathcal{G}_1(x) \exp \mathcal{G}_2(p) \sigma_1 P^-$
 $= 2((x|x)^2 + 4(p|p))(-E_1 + E_2) - ((x|x)^2 + 4(p|p) - 1)P^-$
 $- 2(x|x)(E - 3E_3) + 4F_3^1(p) - 2Q^+(x) + 2Q^-((x|x)x + 2px)$.

Proof. (1) It follows from Proposition 5.7(2), Lemma 6.4(3) and direct calculation.

(2) By Lemma 4.5(1), $a_t \exp(\mathcal{G}_1(x) + \mathcal{G}_1(p))a_t^{-1} = \exp(t([\tilde{A}_3^1(1), \mathcal{G}_1(x)] + [\tilde{A}_3^1(1), \mathcal{G}_2(p)])) = \exp(t(\mathcal{G}_1(x) + 2\mathcal{G}_2(p)))$.

(3) It follows from Proposition 5.7(2), Lemma 6.4(4), and Proposition 4.12(1)(2) and direct calculation. \square

Proof of Main Theorem 2. (1) We consider the element

$$g^{-1} P^- \in \text{Orb}_{F_{4(-20)}}(P^-) = \mathcal{N}_1^-(\mathbf{O}).$$

Since $P^+ = -\sigma_1 P^-$, $(P^- | \sigma_1 g^{-1} P^-) = (g \sigma_1 P^- | P^-) = -(g P^+ | P^-) \neq 0$. Applying Lemma 7.2,

$$(P^- | \sigma_1 g^{-1} P^-) > 0 \quad \text{and} \quad a_{g^{-1} P^-}^G \bar{n}_{g^{-1} P^-}^G g^{-1} P^- = P^-.$$

Now since $(P^- | \sigma_1 g^{-1} P^-) = -(g P^+ | P^-)$, we observe $a_G(g) = a_{g^{-1} P^-}^G$ and $\bar{n}_G(g) = \bar{n}_{g^{-1} P^-}^G$. Thus it follows that

$$(g P^+ | P^-) < 0 \quad \text{and} \quad a_G(g) \bar{n}_G(g) g^{-1} \in (F_{4(-20)})_{P^-}.$$

Hence (i) follows. Put $g_0 = a_G(g) \bar{n}_G(g) g^{-1}$. By Proposition 5.7(1),

$$\psi_3(g_0) \in M \quad \text{and} \quad g_0 = \exp(\mathcal{G}_1(\psi_1(g_0)) + \mathcal{G}_2(\psi_2(g_0))) \psi_3(g_0).$$

Then we see that $m_G(g) = \psi_3(g_0)^{-1} \in M$, and so

$$a_G(g) \bar{n}_G(g) g^{-1} = \exp(\mathcal{G}_1(\psi_1(g_0)) + \mathcal{G}_2(\psi_2(g_0))) m_G(g)^{-1}.$$

Therefore we obtain

$$g = m_G(g) \exp(-\mathcal{G}_1(\psi_1(g_0)) - \mathcal{G}_2(\psi_2(g_0))) a_G(g) \bar{n}_G(g).$$

Next by Lemma 7.5(2), we observe that

$$\exp(-\mathcal{G}_1(\psi_1(g_0)) - \mathcal{G}_2(\psi_2(g_0))) a_G(g) = a_G(g) n_G(G).$$

Therefore we have

$$g = m_G(g) a_G(g) n_G(g) \bar{n}_G(g).$$

Hence (ii) follows.

(2) By Proposition 4.6(3), we note $MAN^+ \sigma_1 N^- = MAN^+ \sigma_1^2 N \sigma_1 = MAN^+ \sigma_1$. Similarly to (1), we consider the element $g^{-1} P^- \in \mathcal{N}_1^-(\mathbf{O})$. Then $(g^{-1} P^- | E_1) = (g E_1 | P^-) \neq 0$. Applying Lemma 7.3(2),

$$(g^{-1} P^- | E_1) < 0 \quad \text{and} \quad \exp\left(\frac{1}{2} \log(-(g^{-1} P^- | E_1))\right) \tilde{A}_3^1(1) X = \sigma_1 P^-.$$

Since $(g^{-1} P^- | E_1) = (g E_1 | P^-)$, $\exp(\frac{1}{2} \log(-(g^{-1} P^- | E_1)) \tilde{A}_3^1(1)) = a'(g)$. Therefore

$$(g E_1 | P^-) < 0 \quad \text{and} \quad \sigma_1 a'(g) g^{-1} P^- = P^-.$$

Hence (i) follows. Put $g_1 = \sigma_1 a'(g) g^{-1}$. By Proposition 5.7(1),

$$\psi_3(g_1) \in M \quad \text{and} \quad g_1 = \exp(\mathcal{G}_1(\psi_1(g_1)) + \mathcal{G}_2(\psi_2(g_1))) \psi_3(g_1).$$

Then we see that $m'(g) = \psi_3(g_1)^{-1} \in M$, and so

$$\sigma_1 a'(g) g^{-1} = \exp(\mathcal{G}_1(\psi_1(g_1)) + \mathcal{G}_2(\psi_2(g_1))) m'(g)^{-1}.$$

Therefore we obtain

$$g = m'(g) \exp(-\mathcal{G}_1(\psi_1(g_1)) - \mathcal{G}_2(\psi_2(g_1))) \sigma_1 a'(g)^{-1}.$$

Since $\sigma_1 a'(g)^{-1} = \tilde{\sigma}_1 (a'(g)^{-1}) \sigma_1 = a'(g) \sigma_1$,

$$g = m'(g) \exp(-\mathcal{G}_1(\psi_1(g_1)) - \mathcal{G}_2(\psi_2(g_1))) a'(g) \sigma_1.$$

Next by Lemma 7.5(2), we observe that

$$\exp(-\mathcal{G}_1(\psi_1(g_1)) - \mathcal{G}_2(\psi_2(g_1))) a'(g) = a'(g) n'(g).$$

Therefore we have

$$g = m'(g) a'(g) n'(g) \sigma_1.$$

Hence (ii) follows.

(3) Denote $S_1 = \{g \in F_{4(-20)} \mid (g P^+ | P^-) \neq 0\}$ and $S_2 = \{g \in F_{4(-20)} \mid (g P^+ | P^-) = 0\}$. Then $F_{4(-20)} = S_1 \coprod S_2$, and by (1)(i),

$$S_1 = \{g \in F_{4(-20)} \mid (g P^+ | P^-) < 0\}.$$

And then the identity element $1 \in S_1$ and $\sigma_1 \in S_2$, and so $S_i \neq \emptyset$.

(Step 1) We will show $MAN^+ N^- = S_1$. Let $g \in S_1$. By (1)(i), $g \in MAN^+ N^-$, and so $S_1 \subset MAN^+ N^-$. Conversely, let $ma_t n \bar{n} \in MAN^+ N^-$ where $m \in M$, $a_t = \exp(t \tilde{A}_3^1(1)) \in A$ ($t \in \mathbb{R}$), $n \in N^+$

and $\bar{n} \in N^-$. By Lemma 4.6(3), $\bar{n} = \sigma_1 n_0 \sigma_1$ for some $n_0 \in N^+$. Since $P^+ = -\sigma_1 P^-$, Lemma 7.5(1) and direct calculation,

$$\begin{aligned} (ma_t n \bar{n} P^+ | P^-) &= -(\sigma_1 n_0 P^- | (ma_t n)^{-1} P^-) \\ &= e^{-2t} (P^+ | P^-) = -4e^{-2t} \neq 0. \end{aligned}$$

Thus $ma_t n \bar{n} \in S_1$, and so $MAN^+ N^- \subset S_1$. Hence $S_1 = MAN^+ N^-$.

(Step 2) We will show $MAN^+ \sigma_1 = S_2$. Let $g \in S_2$. By (2)(ii), $g \in MAN^+ \sigma_1$, and so $S_2 \subset MAN^+ \sigma_1$. Conversely, let $man\sigma_1 \in MAN^+ \sigma_1$ where $m \in M$, $a_t = \exp(t\tilde{A}_3^1(1))$ with $t \in \mathbb{R}$ and $n \in N^+$. Since $P^+ = -\sigma_1 P^-$ and Lemma 7.5(1),

$$(ma_t n \sigma_1 P^+ | P^-) = -(ma_t n P^- | P^-) = -e^{2t} (P^- | P^-) = 0.$$

Thus $g \in S_2$, and so $MAN^+ \sigma_1 \subset S_2$. Therefore $S_2 = MAN^+ \sigma_1 = MAN^+ \sigma_1 N^-$. Hence (3) follows.

(4) We consider the function f on $F_{4(-20)}$: $f(g) = (gP^+ | P^-)$. Since $MAN^+ \sigma_1 N^- = \{g \in F_{4(-20)} \mid f(g) = 0\}$, $MAN^+ \sigma_1 N^-$ is a close set. We will show $MAN^+ \sigma_1 N^-$ has no interior points. Suppose that there exists an open set U of $F_{4(-20)}$ such that $U \subset MAN^+ \sigma_1 N^-$. By (ii), $f(g) = 0$ for all $g \in U$. Now $F_{4(-20)}$ is a real analytic manifold, f is a real analytic function on $F_{4(-20)}$, and $F_{4(-20)}$ is connected (cf. [40, Theorem 2.14.1]). Therefore $f(g) = 0$ for all $g \in F_{4(-20)}$. It contradicts with $\{g \in F_{4(-20)} \mid f(g) \neq 0\} = MAN^+ N^- \neq \emptyset$ by (i). Therefore $MAN^+ \sigma_1 N^-$ has no interior points. Therefore $MAN^+ N^- = (MAN^+ \sigma_1 N^-)^c$ is open dense in $F_{4(-20)}$. \square

Let us define the equivalence relation \sim on $\mathcal{N}_1^-(\mathbf{O})$ as

$$X \sim Y \quad \stackrel{\text{def}}{\iff} \quad Y = rX \quad \text{for some } r > 0$$

for $X, Y \in \mathcal{N}_1^-(\mathbf{O})$, and denote the quotient set:

$$\mathcal{F} := \mathcal{N}_1^-(\mathbf{O}) / \sim.$$

The equivalence class of $X \in \mathcal{N}_1^-(\mathbf{O})$ is denoted by $[X]$. Then we can observe that $F_{4(-20)}$ acts on \mathcal{F} :

$$g[X] := [gX] \quad \text{for } g \in F_{4(-20)} \text{ and } X \in \mathcal{N}_1^-(\mathbf{O}).$$

For $[X] \in \mathcal{F}$, let us denote the stabilizer of $[X]$ as

$$\text{Stab}_{F_{4(-20)}}([X]) := \{g \in F_{4(-20)} \mid g[X] = [X]\}.$$

Theorem 7.6.

- (1) $\text{Stab}_{F_{4(-20)}}([P^-]) = MAN^+$.
- (2) $F_{4(-20)} / (MAN^+) \simeq \mathcal{F}$.

Proof. (1) Let $g = ma_t n^+ \in MAN^+$ where $m \in M$, $a_t = \exp(t\tilde{A}_3^1(1))$ with $t \in \mathbb{R}$ and $n^+ \in N^+$. By Lemma 7.5(1),

$$[gP^-] = [ma_t n^+ P^-] = [e^{2t} P^-] = [P^-].$$

Therefore $g \in \text{Stab}_{F_{4(-20)}}([P^-])$, and so $MAN^+ \subset \text{Stab}_{F_{4(-20)}}([P^-])$.

Conversely, take $g \in \text{Stab}_{F_{4(-20)}}([P^-])$. By Main Theorem 2,

$$(i) \ g \in MAN^+N^- \quad \text{or} \quad (ii) \ g \in MAN^+\sigma_1.$$

Case (i). By Lemma 4.6(3), g can be expressed by

$$g = ma_t n \sigma_1 \exp \mathcal{G}_1(x) \exp \mathcal{G}_2(p) \sigma_1$$

for some $x \in \mathbf{O}$, $p \in \text{Im} \mathbf{O}$, $m \in M$, $a_t = \exp(t\tilde{A}_3^1(1))$ with $t \in \mathbb{R}$ and $n \in N^+$. By Lemma 7.5(3),

$$\begin{aligned} & \sigma_1 \exp \mathcal{G}_1(x) \exp \mathcal{G}_2(p) \sigma_1 P^- \\ = & 2((x|x)^2 + 4(p|p))(-E_1 + E_2) - ((x|x)^2 + 4(p|p) - 1)P^- \\ & - 2(x|x)(E - 3E_3) + 4F_3^1(p) - 2Q^+(x) + 2Q^-(x|x)x + 2px. \end{aligned}$$

Therefore since $gP^- = sP^-$ for some $s > 0$, $x = p = 0$ and so $g = ma_t n \in MAN^+$.

Case (ii). g can be expressed by $g = ma_t n \sigma_1$ where $m \in M$, $a_t = \exp(t\tilde{A}_3^1(1))$ with $t \in \mathbb{R}$ and $n \in N^+$. Since $gP^- = sP^-$ for some $s > 0$, Lemma 7.5(1) and direct calculation,

$$\begin{aligned} 0 &= s(P^-|P^-) = (gP^-|P^-) = (\sigma_1 P^-|(ma_t n)^{-1}P^-) \\ &= e^{-2t}(\sigma_1 P^-|P^-) = 4e^{-2t} \neq 0. \end{aligned}$$

Therefore it is a contradiction and so $g \notin MAN^+\sigma_1$.

Consequently, $g \in MAN^+$ and so $\text{Stab}_{F_{4(-20)}}([P^-]) \subset MAN^+$. Hence $\text{Stab}_{F_{4(-20)}}([P^-]) = MAN^+$.

(2) It follows from Proposition 1.8(2) and (1). \square

8. THE IWASAWA DECOMPOSITION OF $F_{4(-20)}$.

By Propositions 1.6(2) and 3.16(2),

$$\mathcal{H}(\mathbf{O}) \simeq F_{4(-20)}/K$$

We consider AN^+ -orbits on $\mathcal{H}(\mathbf{O})$ to give the Iwasawa decomposition.

Lemma 8.1. *For all $X \in \mathcal{H}(\mathbf{O})$, $(P^-|X) < 0$.*

Proof. By Proposition 1.6(2), $X \in \text{Orb}_{F_{4(-20)}}(E_1)$, and so X can be expressed by $X = gE_1$ for some $g \in F_{4(-20)}$. Then $(P^-|X) = (g^{-1}P^-|E_1)$. Because of $g^{-1}P^- \in \mathcal{N}_1^-(\mathbf{O})$, $(P^-|X) < 0$. \square

For all $X \in \mathcal{H}(\mathbf{O})$, let us define

$$\begin{aligned} a_X &:= \exp\left(\frac{1}{2} \log(-(P^-|X))\tilde{A}_3^1(1)\right) \in A, \\ n_X &:= \exp\left(\mathcal{G}_1\left(\frac{(X)_{F_1^1} - \overline{(X)_{F_2^1}}}{(P^-|X)}\right) + \mathcal{G}_2\left(\frac{\text{Im}((X)_{F_3^1})}{(P^-|X)}\right)\right) \in N^+. \end{aligned}$$

Then by Lemma 8.1, the elements $a_X \in A$ and $n_X \in N^+$ are well-defined, respectively.

Lemma 8.2.

$$(1) (\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{H}(\mathbf{O}) = \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{H}(\mathbf{O}).$$

$$(2) \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}(\mathbf{O}) = \left\{ - \left(s + \sqrt{s^2 + \frac{1}{4}} \right) (-E_1 + E_2) + sP + \frac{1}{2}(E - E_3) \mid s \in \mathbb{R} \right\}.$$

Proof. (1) Obviously, $\mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{H}(\mathbf{O}) \subset (\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{H}(\mathbf{O})$. Conversely, take $X \in (\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{H}(\mathbf{O})$. Then X can be expressed by

$$X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1^1(\bar{x}) + F_2^1(x) + F_3^1(y)$$

for some $\xi_i \in \mathbb{R}$ and $x, y \in \mathbf{O}$. Since $X \in \mathcal{H}(\mathbf{O})$ and Lemma 1.3(2),

$$\begin{aligned} (i) \quad & \xi_1 = (X|E_1) \geq 1, & (ii) \quad & 1 = \text{tr}(X) = \xi_1 + \xi_2 + \xi_3, \\ (iii) \quad & 0 = (X^{\times 2})_{E_1} = \xi_2 \xi_3 - (x|x), \\ (iv) \quad & 0 = (X^{\times 2})_{E_2} = \xi_3 \xi_1 + (x|x), \\ (v) \quad & 0 = (X^{\times 2})_{E_3} = \xi_1 \xi_2 + (y|y). \end{aligned}$$

By (iii)+(iv), $\xi_3(\xi_1 + \xi_2) = 0$. Because of $\xi_3(\xi_1 + \xi_2) = 0$ and $(\xi_1 + \xi_2) + \xi_3 = 1$, $(\xi_1 + \xi_2, \xi_3) = (1, 0)$ or $(0, 1)$. If $\xi_3 = 1$ then $1 \leq \xi_1 = -(x|x) \leq 0$ by (i) and (iv). Therefore $\xi_3 \neq 1$, and so $(\xi_1 + \xi_2, \xi_3) = (1, 0)$. Then by (iii), $(x|x) = 0$, iff $x = 0$. Therefore $X = \xi_1 E_1 + \xi_2 E_2 + F_3^1(y)$, and so $X \in \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{H}(\mathbf{O})$. Thus $(\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{H}(\mathbf{O}) \subset \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{H}(\mathbf{O})$. Hence (1) follows.

(2) Put $\mathcal{P} = \left\{ - \left(s + \sqrt{s^2 + \frac{1}{4}} \right) (-E_1 + E_2) + sP^- + \frac{1}{2}(E - E_3) \mid s \in \mathbb{R} \right\}$. For all $X \in \mathcal{P}$, $X \in \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}(\mathbf{O})$ follows from direct calculation, so that $\mathcal{P} \subset \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}(\mathbf{O})$. Conversely, take $X \in \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}(\mathbf{O})$. By Lemma 6.3(3), X can be expressed by

$$X = r(-E_1 + E_2) + sP^- + t(E - E_3)$$

for some $r, s, t \in \mathbb{R}$. Since $X \in \mathcal{H}(\mathbf{O})$,

$$\begin{aligned} (i) \quad & -r - s + t = (X|E_1) \geq 1, & (ii) \quad & 1 = \text{tr}(X) = 2t, \\ (iii) \quad & 0 = (X^{\times 2})_{E_3} = t^2 - r^2 - 2rs. \end{aligned}$$

By (ii), $t = \frac{1}{2}$ so that by (iii), $4r^2 + 8rs - 1 = 0$. Therefore $r = -s \pm \sqrt{s^2 + \frac{1}{4}}$. Because of (i) and $t = \frac{1}{2}$, $r + s \leq -\frac{1}{2}$. Therefore $r = -s - \sqrt{s^2 + \frac{1}{4}}$, and so

$$X = - \left(s + \sqrt{s^2 + \frac{1}{4}} \right) (-E_1 + E_2) + sP^- + \frac{1}{2}(E - E_3) \in \mathcal{P}.$$

Thus $\mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}(\mathbf{O}) \subset \mathcal{P}$. Hence $\mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}(\mathbf{O}) = \mathcal{P}$. \square

Lemma 8.3. *Assume $X \in \mathcal{H}(\mathbf{O})$, then $a_X n_X X = E_1$. Especially $\mathcal{H}(\mathbf{O}) = \text{Orb}_{AN^+}(E_1)$.*

Proof. Put $n_1 = \exp \mathcal{G}_1 \left(\frac{(X)_{F_1^1} - \overline{(X)_{F_2^1}}}{(P^-|X)} \right) \in N^+$. By Lemma 6.4(1),

- (i) $(P^-|n_1 X) = (P^-|X) \neq 0$,
- (ii) $n_1 X \in \mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})$,
- (iii) $\text{Im}((n_1 X)_{F_3^1}) = \text{Im}((X)_{F_3^1})$.

Because of $n_1 X \in \mathcal{H}(\mathbf{O})$, $n_1 X \in (\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{H}(\mathbf{O})$. Applying Lemma 8.2(1),

$$n_1 X \in \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{H}(\mathbf{O}).$$

Put $n_2 = \exp \mathcal{G}_2 \left(\frac{\text{Im}((n_1 X)_{F_3^1})}{(P^-|n_1 X)} \right) \in N^+$. Then by (i) and (iii), we get

$n_2 = \exp \mathcal{G}_2 \left(\frac{\text{Im}((X)_{F_3^1})}{(P^-|X)} \right)$, and so by Lemma 4.6(1), $n_2 n_1 = n_X \in N^+$.

Using Lemma 6.4(2), $n_X X \in \mathcal{H}(\mathbf{O})$ and (ii),

$$n_X X \in \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}(\mathbf{O}) \quad \text{and} \quad (P^-|n_X X) = (P^-|X).$$

Applying Lemma 8.2(2),

$$n_X X = - \left(s + \sqrt{s^2 + \frac{1}{4}} \right) (-E_1 + E_2) + sP^- + \frac{1}{2}(E - E_3)$$

for some $s \in \mathbb{R}$. Put $c = (P^-|X)$. Then $a_X = \exp(\frac{1}{2} \log(-c) \tilde{A}_3^1(1))$. Because of $(P^-|n_X X) = (P^-|X)$ and Lemma 1.4(i),

$$c = (P^-|n_X X) = 2\{n_X X\}_{-E_1+E_2} = -2 \left(s + \sqrt{s^2 + \frac{1}{4}} \right)$$

and so $c^2 + 4cs - 1 = 0$. Therefore because of Lemma 6.4(3),

$$\begin{aligned} a_X n_X X &= \frac{c}{2} e^{-\log(-c)} (-E_1 + E_2) + \left(\frac{c}{2} \sinh(\log(-c)) + s e^{\log(-c)} \right) P^- \\ &\quad + \frac{1}{2}(E - E_3) \\ &= -\frac{1}{2}(-E_1 + E_2) - \frac{1}{4}(c^2 + 4cs - 1)P^- + \frac{1}{2}(E - E_3) = E_1. \end{aligned}$$

Hence the assertion follows. \square

Proof of Main Theorem 3. We consider the element $g^{-1}E_1 \in \mathcal{H}(\mathbf{O})$. By Lemma 8.1, $(gP^-|E_1) = (P^-|g^{-1}E_1) < 0$ and we observe that

$$a_{g^{-1}E_1} = \exp H(g) \quad \text{and} \quad n_{g^{-1}E_1} = n(g).$$

Now by Lemma 8.3, $a_{g^{-1}E_1} n_{g^{-1}E_1} g^{-1}E_1 = E_1$, and so

$$(\exp H(g)) n(g) g^{-1}E_1 = E_1.$$

Because of $K = (F_{4(-20)})_{E_1}$, it implies that

$$(\exp H(g))n(g)g^{-1} \in (F_{4(-20)})_{E_1} = K.$$

Thus because of $k(g) = ((\exp H(g))n(g)g^{-1})^{-1}$, we obtain

$$k(g) \in K \quad \text{and} \quad g = k(g)((\exp H(g))n(g) \in KAN^+.$$

Hence (2) follows. \square

Lemma 8.4. *Let $\bar{n} = \exp(\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x)) \in N^-$ and $a_t = \exp(t\tilde{A}_3(1))$ where $p \in \text{Im } \mathbf{O}$, $x \in \mathbf{O}$ and $t \in \mathbb{R}$.*

$$(1) \quad H(a_t \bar{n}) = \frac{1}{2} \log(e^{-2t}((e^{2t} + (x|x))^2 + 4(p|p)))\tilde{A}_3^1(1).$$

$$(2) \quad H(\bar{n}) = \frac{1}{2} \log((1 + (x|x))^2 + 4(p|p))\tilde{A}_3^1(1).$$

Proof. (1) By Lemma 4.6(3), $\bar{n} = \sigma_1 \exp \mathcal{G}_2(p) \exp \mathcal{G}_1(x) \sigma_1$, so that

$$(a_t \bar{n} P^- | E_1) = (\exp \mathcal{G}_1(x) \sigma_1 P^- | \exp \mathcal{G}_2(-p) \sigma_1 a_{-t} E_1).$$

Since Lemma 6.4(4), Proposition 4.12(2) and direct calculation,

$$\begin{aligned} \exp \mathcal{G}_1(x) \sigma_1 P^- &= 2(-E_1 + E_2) + ((x|x)^2 - 1)P^- - 2(x|x)(E - 3E_3) \\ &\quad - 2(x|x)Q^+(x) - 2Q^-(x). \end{aligned}$$

Since $E_1 = -\frac{1}{2}((-E_1 + E_2) - (E - E_3))$, Lemma 6.4(3)(4), Proposition 4.12(1) and direct calculation,

$$\begin{aligned} \exp \mathcal{G}_2(-p) \sigma_1 a_{-t} E_1 &= -\frac{1}{2}(e^{-2t}(-E_1 + E_2) + (2e^{-2t}(p|p) + \sinh(2t))P^- \\ &\quad - (E - E_3) - 2e^{-2t}F_3^1(p)). \end{aligned}$$

Since $(-E_1 + E_2 | -E_1 + E_2) = (-E_1 + E_2 | P^-) = (P^- | -E_1 + E_2) = 2$, $(u_1 E + v_1 E_3 | u_2 E + v_2 E_3) = 3u_1 u_2 + u_1 v_2 + u_2 v_1 + v_1 v_2$ and $(P_{12}^\pm(x) | -E_1 + E_2) = (P_{12}^\pm(x) | P^-) = (P_{12}^\pm(x) | E) = (P_{12}^\pm(x) | E_3) = (P_{12}^\pm(x) | F_3^1(p)) = 0$,

$$\begin{aligned} (a_t \bar{n} P^- | E_1) &= -e^{-2t}((x|x)^2 + 2e^{2t}(x|x) + e^{4t} + 4(p|p)) \\ &= -e^{-2t}((e^{2t} + (x|x))^2 + 4(p|p)). \end{aligned}$$

Therefore (1) follows from the definition of H , and (2) follows from (1) and $t = 0$. \square

Let \mathfrak{a}^* be the dual of \mathfrak{a} and $\mathfrak{a}_{\mathbb{C}}^*$ the complexification of \mathfrak{a}^* . We recall $\alpha \in \mathfrak{a}^* \subset \mathfrak{a}_{\mathbb{C}}^*$ satisfies $[H, \mathcal{G}_1(x)] = \alpha(H)\mathcal{G}_1(x)$ for $H \in \mathfrak{a}$ and $\alpha(\tilde{A}_3^1(1)) = 1$. For $\lambda \in \mathfrak{a}^*$, let us define the element $H_\lambda \in \mathfrak{a}$ as $B(H_\lambda, H) = \lambda(H)$ for all $H \in \mathfrak{a}$, and define the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}_{\mathbb{C}}^*$ by setting $\langle \lambda_1, \lambda_2 \rangle = B(H_{\lambda_1}, H_{\lambda_2})$ and extending it to the whole of $\mathfrak{a}_{\mathbb{C}}^*$ by linearity. For any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\lambda_\alpha \in \mathbb{C}$ is defined by

$$\lambda_\alpha := 2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

Because of $\dim \mathfrak{a}_{\mathbb{C}}^* = \dim \mathfrak{a} = 1$, $\lambda = \frac{1}{2}\lambda_{\alpha}\alpha$. Put $m_{\alpha} := \dim \mathfrak{g}_{\alpha}$ and $m_{2\alpha} := \dim \mathfrak{g}_{2\alpha}$. By Lemma 4.5(1), $m_{\alpha} = \dim \mathbf{O} = 8$ and $m_{2\alpha} = \dim (\operatorname{Im} \mathbf{O}) = 7$. Moreover, let us define $\rho \in \mathfrak{a}_{\mathbb{C}}^*$ as

$$\rho := \frac{1}{2}((\dim \mathfrak{g}_{\alpha})\alpha + (\dim \mathfrak{g}_{2\alpha})2\alpha) = \frac{1}{2}(m_{\alpha} + 2m_{2\alpha})\alpha.$$

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Let us consider the *spherical function* φ_{λ} on $F_{4(-20)}$ and the *c-fiction of Harish-Chandra* on $\mathfrak{a}_{\mathbb{C}}^*$. By [12] (cf. [27, Lemma 4.9]), φ_{λ} is given by

$$(8.1) \quad \varphi_{\lambda}(g) := \int_K e^{(\lambda-\rho)(H(gk))} dk = \int_{N^-} e^{(\lambda-\rho)(H(g\bar{n}))} e^{-(\lambda+\rho)(H(\bar{n}))} d\bar{n}$$

for $g \in F_{4(-20)}$, and by [12], the function c is given by

$$(8.2) \quad c(\lambda) := \int_{N^-} e^{-(\lambda+\rho)(H(\bar{n}))} d\bar{n}.$$

Here the measure dk on compact group K is normalized such that the total measure is 1, and the Haar measures of nilpotent groups N and N^- are normalized such that

$$\tilde{\sigma}_1(dn) = d\bar{n} \quad \text{and} \quad \int_{N^-} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1.$$

Proposition 8.5. *Assume $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,*

$$a = \frac{1}{4}(m_{\alpha} + 2m_{2\alpha} + \lambda_{\alpha}), \quad b = \frac{1}{4}(m_{\alpha} + 2m_{2\alpha} - \lambda_{\alpha}),$$

and $a_t = \exp(t\tilde{A}_3(1))$ with $t \in \mathbb{R}$. Then there exists the constant $C_0 \in \mathbb{R}$ such that

$$\begin{aligned} \text{(i)} \quad c(\lambda) &= C_0 \int_{\mathbb{R}^{m_{\alpha}} \times \mathbb{R}^{m_{2\alpha}}} ((1 + (x|x))^2 + 4(p|p))^{-a} dx dp, \\ \text{(ii)} \quad \varphi_{\lambda}(a_t) &= C_0 \int_{\mathbb{R}^{m_{\alpha}} \times \mathbb{R}^{m_{2\alpha}}} e^{2bt} ((e^{2t} + (x|x))^2 + 4(p|p))^{-b} \cdot \\ &\quad ((1 + (x|x))^2 + 4(p|p))^{-a} dx dp \end{aligned}$$

where the measure dx and dp are the Euclidean measure.

Proof. (i) follows from (8.2) and Lemma 8.4(2), and (ii) follows from (8.1) and Lemma 8.4(1). \square

Corollary 8.6. ([12]) $e^{2bt}\varphi_{\lambda}(a_t) \rightarrow c(\lambda)$ ($t \rightarrow +\infty$).

Proof. By Proposition 8.5, $e^{2bt}\varphi_{\lambda}(a_t) = C_0 \int_{\mathbb{R}^{m_{\alpha}} \times \mathbb{R}^{m_{2\alpha}}} ((1 + e^{-2t}(x|x))^2 + 4e^{-4t}(p|p))^{-b} ((1 + (x|x))^2 + 4(p|p))^{-a} dx dp \rightarrow c(\lambda)$ ($t \rightarrow +\infty$). \square

Let us denote $Q(\phi) := -\langle \alpha, \alpha \rangle B(\phi, \tilde{\sigma}_1\phi)$ for $\phi \in \mathfrak{f}_{4(-20)}$. Since Lemma 4.2(2) and direct calculation, we have:

Proposition 8.7. *Assume $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $p \in \operatorname{Im} \mathbf{O}$ and $x \in \mathbf{O}$. Then $Q(\mathcal{G}_{-1}(x)) = 2(x|x)$ and $Q(\mathcal{G}_{-2}(p)) = 2(p|p)$.*

Corollary 8.8. ([14], [31], cf. [26, Lemma 4.12]) *Assume $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then $e^{\lambda(H(\exp(X+Y)))} = ((1 + \frac{1}{2}Q(X))^2 + 2Q(Y))^{\frac{\lambda\alpha}{4}}$ for $X \in \mathfrak{g}_{-\alpha}$ and $Y \in \mathfrak{g}_{-2\alpha}$.*

Proof. It follows from Proposition 8.7 and Lemma 8.4(2). \square

Remark 8.9. ([14], [31], cf. [28], [30]) By Lemma 8.5(1)(i), changing variables to polar coordinates, up to the constant multiple, $c(\lambda)$ is equal to

$$\begin{aligned} & \int_0^\infty \int_0^\infty t^{m_\alpha-1} s^{m_{2\alpha}-1} ((1+t^2)^2 + s^2)^{-\frac{1}{4}(\lambda_\alpha+m_\alpha+2m_{2\alpha})} ds dt \\ &= \int_0^\infty \int_0^\infty \frac{(\frac{s}{1+t^2})^{m_{2\alpha}-1}}{(1+(\frac{s}{1+t^2})^2)^{\frac{1}{4}(\lambda_\alpha+m_\alpha+2m_{2\alpha})}} \cdot \frac{t^{m_\alpha-1}}{(1+t^2)^{\frac{1}{2}(\lambda_\alpha+m_\alpha)+1}} ds dt \\ &= \int_0^\infty \frac{u^{m_{2\alpha}-1}}{(1+u^2)^{\frac{1}{4}(\lambda_\alpha+m_\alpha+2m_{2\alpha})}} du \cdot \int_0^\infty \frac{t^{m_\alpha-1}}{(1+t^2)^{\frac{1}{2}(\lambda_\alpha+m_\alpha)}} dt \end{aligned}$$

By using the integral formula $\int_0^\infty \frac{x^a}{(1+x^c)^{b+1}} dx = \frac{1}{c} \frac{\Gamma((a+1)/c) \Gamma[b - ((a-c+1)/c)]}{\Gamma(1+b)}$ ($\text{Re}(c) > 0$; $\text{Re}(b) > -1$; $\text{Re}(b) > \text{Re}((a-c+1)/c)$), up to the constant multiple, this integral is equal to

$$\frac{\Gamma(\frac{\lambda_\alpha}{2})}{\Gamma(\frac{\lambda_\alpha+m_\alpha}{2})} \frac{\Gamma(\frac{\lambda_\alpha+m_\alpha}{4})}{\Gamma(\frac{\lambda_\alpha+m_\alpha}{4} + \frac{m_{2\alpha}}{2})}.$$

This calculation implies the *Gindikin and Karpelevich formula* of $F_{4(-20)}$ which is known [5] (cf. [28, (4.3)], [19]).

Remark 8.10. Note that the Iwasawa decomposition of $F_{4(-20)}$ was studied by R. Takahashi [33, Theorem 1]. Main Theorem 3 gives explicit formulas of $H(g)$ and $n(g)$.

9. THE IWASAWA DECOMPOSITION WITH RESPECT TO K_ϵ .

By Proposition 1.6(3), Lemma 6.2(2) and Proposition 3.16(3),

$$\mathcal{H}'(\mathbf{O}) \simeq F_{4(-20)}/K_\epsilon$$

so we consider AN^+ -orbits on $\mathcal{H}'(\mathbf{O})$ to give the Iwasawa decomposition with respect to K_ϵ . Let us denote the domains D_ϵ and D_ϵ^+ of \mathcal{J}^1 as

$$\begin{aligned} D_\epsilon &:= \{X \in \mathcal{H}'(\mathbf{O}) \mid (P^-|X) \neq 0\}, \\ D_\epsilon^+ &:= \{X \in \mathcal{H}'(\mathbf{O}) \mid (P^-|X) > 0\}. \end{aligned}$$

For all $X \in \mathcal{H}'(\mathbf{O})$, let us denote

$$\begin{aligned} a_X^\epsilon &:= \exp\left(\frac{1}{2} \log((P^-|X)) \tilde{A}_3^1(1)\right) \in A, \\ n_X^\epsilon &:= \exp\left(\mathcal{G}_1 \left(\frac{(X)_{F_1^1} - \overline{(X)_{F_2^1}}}{(P^-|X)} \right) + \mathcal{G}_2 \left(\frac{\text{Im}((X)_{F_3^1})}{(P^-|X)} \right)\right) \in N^+. \end{aligned}$$

If $X \in D_\epsilon$ then the element $n_X^\epsilon \in N^+$ is well-defined, and if $X \in D_\epsilon^+ \subset D_\epsilon$ then the element $a_X^\epsilon \in A$ is well-defined.

Lemma 9.1.

$$(1) (\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{D}_\epsilon = \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{D}_\epsilon.$$

$$(2) \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}'(\mathbf{O}) = \left\{ \left(-s + \sqrt{s^2 + \frac{1}{4}} \right) (-E_1 + E_2) + sP + \frac{1}{2}(E - E_3) \mid s \in \mathbb{R} \right\}.$$

Proof. (1) Obviously $\mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{D}_\epsilon \subset (\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{D}_\epsilon$. Conversely, take $X \in (\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{D}_\epsilon$. Then X can be expressed by

$$X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1^1(\bar{x}) + F_2^1(x) + F_3^1(y)$$

with $\xi_i \in \mathbb{R}$ and $x, y \in \mathbf{O}$. Since $X \in \mathcal{H}'(\mathbf{O})$ and Lemma 1.3(2),

$$\begin{aligned} (i) \quad & \xi_1 = (X|E_1) \leq 0, & (ii) \quad & 1 = \text{tr}(X) = \xi_1 + \xi_2 + \xi_3, \\ (iii) \quad & 0 = (X^{\times 2})_{E_1} = \xi_2 \xi_3 - (x|x), & (iv) \quad & 0 = (X^{\times 2})_{E_2} = \xi_3 \xi_1 + (x|x), \\ (v) \quad & 0 = (X^{\times 2})_{E_3} = \xi_1 \xi_2 + (y|y), & (vi) \quad & 0 = (X^{\times 2})_{F_3^1} = (x|x) - \xi_3 y. \end{aligned}$$

By (iii)+(iv), $\xi_3(\xi_1 + \xi_2) = 0$. Because of $\xi_3(\xi_1 + \xi_2) = 0$ and $(\xi_1 + \xi_2) + \xi_3 = 1$, $(\xi_1 + \xi_2, \xi_3) = (1, 0)$ or $(0, 1)$. Suppose that $(\xi_1 + \xi_2, \xi_3) = (0, 1)$. Then by (iii), (iv) and (vi), $X = (x|x)P^- + E_3 + Q^+(x)$ and so $(P^-|X) = 0$. It contradicts with $X \in \mathcal{D}_\epsilon$. Therefore $(\xi_1 + \xi_2, \xi_3) = (1, 0)$. Then by (iv), $(x|x) = 0$ iff $x = 0$. Thus $X = \xi_1 E_1 + \xi_2 E_2 + F_3^1(y)$, and so $X \in \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{D}_\epsilon$. Therefore $(\mathcal{J}^1(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{D}_\epsilon \subset \mathcal{J}^1(2; \mathbf{O}) \cap \mathcal{D}_\epsilon$. Hence (1) follows.

(2) Put $\mathcal{P} = \left\{ \left(-s + \sqrt{s^2 + \frac{1}{4}} \right) (-E_1 + E_2) + sP^- + \frac{1}{2}(E - E_3) \mid s \in \mathbb{R} \right\}$. For all $X \in \mathcal{P}$, by direct calculation, $X \in \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}'(\mathbf{O})$, and so $\mathcal{P} \subset \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}'(\mathbf{O})$. Conversely, take $X \in \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}'(\mathbf{O})$. Then by Lemma 6.3(3), X can be expressed by

$$X = r(-E_1 + E_2) + sP^- + t(E - E_3)$$

with $r, s, t \in \mathbb{R}$. Because of $X \in \mathcal{H}'(\mathbf{O})$,

$$(i) \quad -r - s + t \leq 0, \quad (ii) \quad 1 = \text{tr}(X) = 2t, \quad (iii) \quad 0 = (X^{\times 2})_{E_3} = t^2 - r^2 - 2rs.$$

By (ii), $t = \frac{1}{2}$ and so by (iii), $4r^2 + 8rs - 1 = 0$. Therefore $r = -s \pm \sqrt{s^2 + \frac{1}{4}}$. Because of (i) and $t = \frac{1}{2}$, $r + s \geq \frac{1}{2}$, so that $r = -s + \sqrt{s^2 + \frac{1}{4}}$. Thus

$$X = \left(-s + \sqrt{s^2 + \frac{1}{4}} \right) (-E_1 + E_2) + sP^- + \frac{1}{2}(E - E_3) \in \mathcal{P}.$$

Therefore $\mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}'(\mathbf{O}) \subset \mathcal{P}$. Hence $\mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}'(\mathbf{O}) = \mathcal{P}$. \square

Lemma 9.2.

- (1) For all $X \in \mathcal{D}_\epsilon$, $X \in \mathcal{D}_\epsilon^+$ and $a_X^\epsilon n_X^\epsilon X = E_2$.
 (2) $\mathcal{D}_\epsilon = \mathcal{D}_\epsilon^+ = \text{Orb}_{AN^+}(E_2)$.

Proof. (1) Put $n_1 = \exp \mathcal{G}_1 \left(\frac{(X)_{F_1^1} - \overline{(X)_{F_2^1}}}{(P^-|X)} \right) \in N^+$. by Lemma 6.4(1),

- (i) $(P^-|n_1X) = (P^-|X) \neq 0$, (ii) $n_1X \in \mathcal{J}(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})$,
 (iii) $\text{Im}((n_1X)_{F_3^1}) = \text{Im}((X)_{F_3^1})$.

Since (i), (ii) and $n_1X \in \mathcal{H}'(\mathbf{O})$, $n_1X \in (\mathcal{J}(2; \mathbf{O}) \oplus \mathbb{R}E_3 \oplus Q^+(\mathbf{O})) \cap \mathcal{D}_\epsilon$. Applying Lemma 9.1(1), we get

$$n_1X \in \mathcal{J}(2; \mathbf{O}) \cap \mathcal{D}_\epsilon.$$

Put $n_2 = \exp \mathcal{G}_2 \left(\frac{\text{Im}((n_1X)_{F_3^1})}{(P^-|n_1X)} \right) = \exp \mathcal{G}_2 \left(\frac{\text{Im}((X)_{F_3^1})}{(P^-|X)} \right) \in N^+$. And we see that $n_2n_1 = n_X^\epsilon \in N^+$ by Lemma 4.6(1). Applying Lemma 6.4(2),

$$n_X^\epsilon X \in \mathcal{J}^1(2; \mathbb{R}) \cap \mathcal{H}'(\mathbf{O}) \quad \text{and} \quad (P^-|n_X^\epsilon X) = (P^-|X).$$

Using Lemma 9.1(2),

$$n_X^\epsilon X = \left(-s + \sqrt{s^2 + \frac{1}{4}} \right) (-E_1 + E_2) + sP + \frac{1}{2}(E - E_3)$$

for some $s \in \mathbb{R}$. Put $c = (P^-|X)$. Then because of $(P^-|n_X^\epsilon X) = (P^-|X)$ and Lemma 1.4(i),

$$c = (P^-|n_X^\epsilon X) = 2\{n_X^\epsilon X\}_{-E_1+E_2} = 2 \left(-s + \sqrt{s^2 + \frac{1}{4}} \right) > 0$$

and so $c^2 + 4cs - 1 = 0$. Therefore $(P^-|X) > 0$ follows, and so a_X^ϵ is well-defined. Now $a_X^\epsilon = \exp(\frac{1}{2}(\log c)\tilde{A}_3^1(1))$. Thus since Lemma 6.4(3) and direct calculation,

$$\begin{aligned} a_X^\epsilon n_X^\epsilon X &= \frac{c}{2}e^{-\log c}(-E_1 + E_2) + \left(\frac{c}{2} \sinh(\log c) + se^{\log c} \right) P^- + \frac{E - E_3}{2} \\ &= \frac{1}{2}(-E_1 + E_2) + \frac{1}{4}(c^2 + 4cs - 1)P^- + \frac{1}{2}(E - E_3) = E_2. \end{aligned}$$

Hence (1) follows.

(2) It follows that $\mathcal{D}_\epsilon^+ \subset \mathcal{D}_\epsilon \subset \mathcal{D}_\epsilon^+$ from (1). Hence (2) follows. \square

Proof of Main Theorem 4. (1) Let $g \in \mathcal{D}$. Because of $\mathcal{H}'(\mathbf{O}) = \text{Orb}_{F_4(-20)}(E_2)$, we consider the element $g^{-1}E_2 \in \mathcal{H}'(\mathbf{O})$. Because of $(P^-|g^{-1}E_2) = (gP^-|E_2) > 0$, $g^{-1}E_2 \in \mathcal{D}_\epsilon^+$ and we see that

$$a_{g^{-1}E_2}^\epsilon = \exp H_\epsilon(g) \quad \text{and} \quad n_{g^{-1}E_2}^\epsilon = n_\epsilon(g).$$

Applying Lemma 9.2(2),

$$a_{g^{-1}E_2}^\epsilon n_{g^{-1}E_2}^\epsilon g^{-1}E_2 = E_2.$$

Because of $K_\epsilon = (F_{4(-20)})_{E_2}$, it implies that

$$(\exp H_\epsilon(g))n_\epsilon(g)g^{-1} \in K_\epsilon.$$

Because of $k_\epsilon(g) = ((\exp H_\epsilon(g))n_\epsilon(g)g^{-1})^{-1}$, we obtain

$$k_\epsilon(g) \in K_\epsilon \quad \text{and} \quad g = k_\epsilon(g)(\exp H_\epsilon(g))n_\epsilon(g) \in K_\epsilon AN^+.$$

Hence (1) follows.

(2) At first, $\mathcal{D} \subset K_\epsilon AN^+$ follows from (1). Conversely, take $g \in K_\epsilon AN^+$. Then g can be expressed by $g = k_\epsilon a_t n$ where $k_\epsilon \in K_\epsilon$, $a_t = \exp(t\tilde{A}_3^1(1))$ with $t \in \mathbb{R}$ and $n \in N^+$. Since $K_\epsilon = (F_{4(-20)})_{E_2}$ and Lemma 7.5(1),

$$(gP^-|E_2) = (a_t n P^-|k_\epsilon^{-1}E_2) = e^{2t}(P^-|E_2) = e^{2t} > 0.$$

Thus $g \in \mathcal{D}$, and so $K_\epsilon AN^+ \subset \mathcal{D}$. Therefore $\mathcal{D} = K_\epsilon AN^+$.

At second, since $\mathcal{H}'(\mathbf{O}) = \text{Orb}_{F_{4(-20)}}(E_2)$ and Lemma 9.2(2),

$$\begin{aligned} \mathcal{D} &= \{g \in F_{4(-20)} | (P^-|g^{-1}E_2) > 0\} = \{g \in F_{4(-20)} | g^{-1}E_2 \in \mathcal{D}_\epsilon^+\} \\ &= \{g \in F_{4(-20)} | g^{-1}E_2 \in \mathcal{D}_\epsilon\} = \{g \in F_{4(-20)} | (P^-|g^{-1}E_2) \neq 0\} \\ &= \{g \in F_{4(-20)} | (gP^-|E_2) \neq 0\}. \end{aligned}$$

So when we consider the real-analytic function f on $F_{4(-20)}$: $f(g) = (gP^-|E_2)$, we obtain $\mathcal{D} = \{g \in F_{4(-20)} | f(g) \neq 0\}$ and $\mathcal{D} \neq \emptyset$. Therefore \mathcal{D} is an open and dense set in $F_{4(-20)}$, since $F_{4(-20)}$ is connected. Hence (2) follows. \square

By direct calculation, we can prove the following corollary as similar to Lemma 8.4 and Corollary 8.8.

Corollary 9.3. *Assume $p \in \text{Im}\mathbf{O}$ and $x \in \mathbf{O}$. Let $\bar{n} = \exp(\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x)) \in N^-$.*

$$(1) \ H_\epsilon(\bar{n}) = \frac{1}{2} \log((1 - (x|x))^2 + 4(p|p))\tilde{A}_3^1(1).$$

$$(2) \text{ (cf. [26, Lemma 4.12])}$$

$$\begin{aligned} e^{\lambda(H_\epsilon(\exp(\mathcal{G}_{-2}(p) + \mathcal{G}_{-1}(x))))} &= ((1 - (x|x))^2 + 4(p|p))^{\frac{\lambda\alpha}{4}} \\ &= ((1 - \frac{1}{2}Q(\mathcal{G}_{-1}(x)))^2 + 2Q(\mathcal{G}_{-2}(p)))^{\frac{\lambda\alpha}{4}}. \end{aligned}$$

The group K_ϵ acts on $\mathcal{F} = G/(MAN^+)$. Then K_ϵ -orbits is finite [21, Theorem 1], and K_ϵ -orbits induce the decomposition of G [21, Theorem 3]. Let us denote the element $P' := h^1(-1, 0, 1; 0, 1, 0) \in \mathcal{J}^1$.

Lemma 9.4. *Let $X \in \mathcal{N}_1^-(\mathbf{O})$.*

(1) *Assume that $(X|E_2) \neq 0$. Then there exists $k' \in K_\epsilon$ such that $(k'X)_{F_2^1} = 0$.*

(2) *Assume that $(X|E_2) = 0$. Then there exists $k' \in K_\epsilon$ such that $k'X = rP'$ for some $r > 0$.*

$$(3) \ P^- = \exp(\tilde{A}_1^1(\frac{\pi}{2}))P'.$$

Proof. (1) X can be expressed by $X = X_{\sigma_2} + X_{-\sigma_2}$ where $X_{\sigma_2} \in \mathcal{J}_{\sigma_2}$ and $X_{-\sigma_2} \in \mathcal{J}_{-\sigma_2}$. Then

$$X_{-\sigma_2} = F_3^1(u) + F_1^1(v) \quad \text{and} \quad X_{\sigma_2} = pE_2 + q(E - E_2) + Y$$

where $u, v \in \mathbf{O}$, $p, q \in \mathbb{R}$ and $Y \in (\mathcal{J}^1)_{2E_2, -1}$. And since $K_\epsilon = (F_{4(-20)})_{E_2} = (F_{4(-20)})^{\tilde{\sigma}_2}$, for all $k_0 \in K_\epsilon$, we get $k_0 X_{-\sigma_2} \in \mathcal{J}_{-\sigma_2}$ (cf. Lemma 3.15(2)), $k_0 X_{\sigma_2} = pE_2 + q(E - E_2) + k_0 Y$ and $k_0 Y \in (\mathcal{J}^1)_{2E_2, -1}$. Now $Y \neq 0$. Suppose that $Y = 0$. Then $0 = (X^{\times 2})_{E_2} = q^2$ iff $q = 0$, and so $(X|E_2) = p = \text{tr}(X) = 0$. It contradicts with $(X|E_2) \neq 0$.

Since $Y \neq 0$ and $\{X \in (\mathcal{J}^1)_{2E_2, -1} \mid Q_{E_2}(X) = 1\} = \text{Orb}_{K_\epsilon}(E_1 - E_3) \amalg \text{Orb}_{K_\epsilon}(-E_1 + E_3)$ which is similar to Lemma 3.8(2) is valid, there exists $k' \in K_\epsilon$ and $r > 0$ such that

$$k'Y = r(E_1 - E_3) \quad \text{or} \quad r(-E_1 + E_3),$$

Then it implies that $(k'X)_{F_2^1} = 0$.

(2) X can be expressed by $X = h^1(-r, 0, r; x_1, x_2, x_3)$ where $r > 0$ and $x_i \in \mathbf{O}$. For $j \in \{1, 3\}$, $0 = (X^{\times 2})_{E_j} = -\epsilon_1(j)(x_j|x_j)$ iff $x_j = 0$, because of $X \in \mathcal{H}'(\mathbf{O})$. Thus $X = h^1(-r, 0, r; 0, x_2, 0)$. Using Proposition 3.10(1), there exists $k' \in D_4 \subset K_\epsilon$ such that $k'X = h^1(-r, 0, r; 0, r, 0) = rP'$. Hence (2) follows.

(3) It follows from Lemma 3.7(1). \square

Proposition 9.5.

- (1) $\text{Orb}_{K_\epsilon}([P^-]) = \{[X] \in \mathcal{F} \mid (X|E_2) \neq 0\}$.
- (2) $\text{Orb}_{K_\epsilon}([P']) = \{[X] \in \mathcal{F} \mid (X|E_2) = 0\}$.
- (3) $\mathcal{F} = \text{Orb}_{K_\epsilon}([P^-]) \amalg \text{Orb}_{K_\epsilon}([P'])$.

Proof. (1) Put $\mathcal{O} = \{[X] \in \mathcal{F} \mid (X|E_2) \neq 0\}$. Since $K_\epsilon = (F_{4(-20)})_{E_2}$, K_ϵ acts on \mathcal{O} . We will show that this action is transitive. let $X \in \mathcal{N}_1^-(\mathbf{O})$ where $(X|E_2) \neq 0$. At first, by Lemma 9.4(1), there exists $k' \in K_\epsilon$ such that $(k'X)_{F_2^1} = 0$. Thus $k'X$ can be expressed by

$$k'X = h^1(\xi_1, \xi_2, \xi_3; x_1, 0, x_3) \quad \text{for some } \xi_i \in \mathbb{R} \text{ and } x_i \in \mathbf{O}.$$

Since $k'X \in \mathcal{N}_1^-(\mathbf{O})$,

- (i) $\xi_1 = (k'X|E_1) < 0$, (ii) $\xi_1 + \xi_2 + \xi_3 = \text{tr}(k'X) = 0$,
- (iii) $0 = (k'X)_{E_2} = \xi_3\xi_1$, (iv) $0 = (k'X)_{E_1} = \xi_2\xi_3 - (x_1|x_1)$.

Then it follows that $\xi_3 = 0$, $\xi_1 = -\xi_2$ and $\xi_2 > 0$ form (i), (ii) and (iii). And by (iv), $(x_1|x_1) = 0$ iff $x_1 = 0$. Thus $k'X = h^1(-\xi_2, \xi_2, 0; 0, 0, x_3)$ with $\xi_2 > 0$. At second, using Proposition 3.10(1), there exists $k_0 \in D_4 \subset K_\epsilon$ such that $k_0 k'X = h^1(-\xi_2, \xi_2, 0; 0, 0, \xi_2) = \xi_2 P^-$. Therefore $[X] \in \text{Orb}_{K_\epsilon}([P^-])$, and so the transitivity is proved. Hence (1) follows.

(2) Put $\mathcal{O}' = \{[X] \in \mathcal{F} \mid (X|E_2) = 0\}$. Since $K_\epsilon = (F_{4(-20)})_{E_2}$, K_ϵ acts on \mathcal{O}' . We will show that this action is transitive. Take $X \in \mathcal{N}_1^-(\mathbf{O})$ where $(X|E_2) = 0$. By Lemma 9.4(2), there exists $k' \in K_\epsilon$.

$[k'X] = [P'] \in Orb_{K_\epsilon}([P'])$ and so and so the transitivity is proved. Hence (2) follows.

(3) By (1) and (2), $\mathcal{F} = \mathcal{O} \coprod \mathcal{O}' = Orb_{K_\epsilon}([P^-]) \coprod Orb_{K_\epsilon}([P'])$. \square

Theorem 9.6.

(1)

$$\begin{aligned} K_\epsilon MAN^+ &= \{g \in F_{4(-20)} \mid (gP^-|E_2) \neq 0\} \\ &= \{g \in F_{4(-20)} \mid (gP^-|E_2) > 0\} = K_\epsilon AN^+. \end{aligned}$$

$$(2) \quad K_\epsilon \exp\left(-\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) MAN^+ = \{g \in F_{4(-20)} \mid (gP^-|E_2) = 0\}.$$

$$(3) \quad F_{4(-20)} = K_\epsilon MAN^+ \coprod K_\epsilon \exp\left(-\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) MAN^+.$$

Proof. Denote $\mathcal{D}_1 = \{g \in F_{4(-20)} \mid (gP^-|E_2) \neq 0\}$ and $\mathcal{D}_2 = \{g \in F_{4(-20)} \mid (gP^-|E_2) = 0\}$.

(1) Let $g \in \mathcal{D}_1$. Using Proposition 9.5(1), $[gP^-] \in Orb_{K_\epsilon}([P^-])$. Thus there exists $k' \in K_\epsilon$ such that

$$k'g[P^-] = k'[gP^-] = [P^-].$$

Therefore $k'g \in Stab_{F_{4(-20)}}([P^-])$, and so by Theorem 7.6(1),

$$k'g = man \quad \text{for some } m \in M, a \in A \text{ and } n \in N^+.$$

Hence $g = k^{-1}man \in K_\epsilon MAN$ and so $\mathcal{D}_1 \subset K_\epsilon MAN$.

Conversely, let $k'a_tmn \in K_\epsilon MAN$ where $k' \in K_\epsilon$, $a_t = \exp(t\tilde{A}_3^1(1))$ with $t \in \mathbb{R}$ and $n \in N^-$. Since $k'^{-1} \in (F_{4(-20)})_{E_2}$ and Lemma 7.5(1),

$$(k'a_tmnP^-|E_2) = (ma_tnP^-|k'^{-1}E_2) = e^{2t}(P^-|E_2) = e^{2t} \neq 0.$$

Thus $k'a_tmn \in \mathcal{D}_1$, and so $K_\epsilon MAN \subset \mathcal{D}_1$. Therefore $K_\epsilon MAN = \mathcal{D}_1$. Next by Main Theorem 4(2), $\mathcal{D}_1 = \mathcal{D} = K_\epsilon AN^+$. Hence (1) follows.

(2) Let $g \in \mathcal{D}_2$. Using Proposition 9.5(2), $[gP^-] \in Orb_{K_\epsilon}([P'])$. Thus there exists $k' \in K_\epsilon$ such that $[k'gP^-] = [P']$. Next by Lemma 9.4(3),

$$\exp\left(\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) k'g[P^-] = \left[\exp\left(\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) P'\right] = [P^-].$$

Thus $\exp\left(\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) k'g \in Stab_{F_{4(-20)}}([P^-])$, and so by Theorem 7.6(1),

$$\exp\left(\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) k'g = man \quad \text{for some } m \in M, a \in A \text{ and } n \in N^+.$$

Hence $g = k^{-1} \exp\left(-\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) man \in K_\epsilon \exp\left(-\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) MAN$ and so $\mathcal{D}_2 \subset K_\epsilon \exp\left(-\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) MAN$.

Conversely, let $k' \exp\left(-\tilde{A}_1^1\left(\frac{\pi}{2}\right)\right) a_tmn \in K_\epsilon MAN$ where $k' \in K_\epsilon$, $a_t = \exp(t\tilde{A}_3^1(1))$ with $t \in \mathbb{R}$ and $n \in N^-$. Since $k'^{-1} \in (F_{4(-20)})_{E_2}$,

Lemma 7.5(1) and Lemma 3.7(2),

$$\begin{aligned} (k' \exp \left(-\tilde{A}_1^1 \left(\frac{\pi}{2} \right) \right) a_t m n P^- | E_2) &= (m a_t n P^- | \exp \left(\tilde{A}_1^1 \left(\frac{\pi}{2} \right) \right) k'^{-1} E_2) \\ &= e^{2t} (P^- | \exp \left(\tilde{A}_1^1 \left(\frac{\pi}{2} \right) \right) E_2) = e^{2t} (P^- | E_3) = 0. \end{aligned}$$

Thus $k' \exp \left(-\tilde{A}_1^1 \left(\frac{\pi}{2} \right) \right) a_t m n \in \mathcal{D}_2$, and so $K_\epsilon \exp \left(-\tilde{A}_1^1 \left(\frac{\pi}{2} \right) \right) MAN \subset \mathcal{D}_2$. Therefore $K_\epsilon \exp \left(-\tilde{A}_1^1 \left(\frac{\pi}{2} \right) \right) MAN = \mathcal{D}_2$.

(3) It follows from (1), (2) and $F_{4(-20)} = \mathcal{D}_1 \coprod \mathcal{D}_2$. \square

Remark 9.7. Proposition 9.5(3) and Theorem 9.6 are special cases of [21, Theorems 1 and 3], so the decomposition in Theorem 9.6(3) is called a *Matsuki decomposition* of $F_{4(-20)}$.

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